Journal of Statistical Physics, Vol. 100, Nos. 3/4, 2000

Fokker–Planck Equations as Scaling Limits of Reversible Quantum Systems

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Received October 11, 1999; final March 2, 2000

We consider a quantum particle moving in a harmonic exterior potential and linearly coupled to a heat bath of quantum oscillators. Caldeira and Leggett derived the Fokker–Planck equation with friction for the Wigner distribution of the particle in the large-temperature limit; however, their (nonrigorous) derivation was not free of criticism, especially since the limiting equation is not of Lindblad form. In this paper we recover the correct form of their result in a rigorous way. We also point out that the source of the diffusion is physically restrictive under this scaling. We investigate the model at a fixed temperature and in the large-time limit, where the origin of the diffusion is a cumulative effect of many resonant collisions. We obtain a heat equation with a friction term for the radial process in phase space and we prove the Einstein relation in this case.

KEY WORDS: Fokker–Planck equation; Wigner distribution; scaling limit; coupled harmonic oscillators.

1. INTRODUCTION

In ref. 5, Caldeira and Leggett introduced a Hamiltonian for a quantum system of a test-particle coupled to an abstract reservoir. The Schrödinger equation for the evolution of the quantum state can be equivalently written as a kinetic (phase-space) equation for the associated Wigner distribution of the test particle-reservoir system. The goal of ref. 5 was to derive (formally)

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0022-4715/00/0800-0543$18.00/0

a Fokker–Planck equation for the Wigner distribution of the test-particle by considering various asymptotic regimes which we explain below and by "tracing out" the reservoir coordinates. The Fokker–Planck equation represents an irreversible collisional evolution with a diffusive term, while the Schrödinger equation is reversible. Hence this derivation was expected to shed some light on the origin of diffusion in the evolution of a small system coupled to an infinite reservoir. Caldeira and Leggett used a Feynman path integral approach which has no rigorous mathematical justification (despite its great successes in formal computations). More importantly, several other steps in their derivation admittedly lack mathematical precision.

Starting from this observation, the aim of the present paper is twofold. In Sections 4 and 5 we present a mathematically rigorous derivation of the frictionless Fokker–Planck equation from the model introduced in ref. 5. In Sections 6 and 7 we show how to recover another type of Fokker–Planck equation from the Caldeira–Leggett Hamiltonian, using a different diffusion mechanism, scalings and limiting procedures.

In both models we focus on determining the precise assumptions which lead to the given equations. We do not attempt to describe the variety of physical models for which the Caldeira–Leggett Hamiltonian is used as a phenomenological description. In particular we do not investigate to what extent the required assumptions are realistic in actual applications. However, we keep in mind one possible physical realization of the Caldeira–Leggett dynamics, namely the motion of an electron in a nearest neighbor harmonic crystal (Section 2).

We point out that ref. 5 heavily relies on the use of ideas from Feynman, Hibbs, and Vernon.[25, 26] In particular Feynman and Vernon[26] considered a system of the form $\{$test "particle" $(A)+$ reservoir $(R)\}$. The Hamiltonian is $H_A+H_R+H_I$, where H_A is the free Hamiltonian for the test-particle, H_R is the free Hamiltonian for the reservoir, and H_I is the interaction Hamiltonian. They integrated out the reservoir variables, i.e., they computed the time evolution of the wave function of the test-particle itself, given by $Tr_R\{\exp(it\hbar^{-1}(H_A+H_R+H_I))\}$, where Tr_R is the partial trace on the Hilbert space of the reservoir and $\hbar=h/2\pi$ where h is the Planck constant. Feynman path integral formalism was used which is particularly powerful when H_R is quadratic and the interaction is linear in the reservoir variables. In this case the partial trace Tr_R leads to explicit Gaussian integrals in the reservoir variables, but in general it is not Gaussian in the test-particle variables. However, if the total Hamiltonian is quadratic, in particular the coupling is linear in the test-particle variables, then the full evolution is given by a Gaussian integral, which, in principle, is explicit. The difficulty stems from the large (infinite) number of variables.

In this context ref. 5 introduces the following Hamiltonian,

$$
\begin{aligned}
H_{CL} &= H_A + H_R + H_I \\
&= \left(-\frac{\hbar^2}{2M}\Delta_x + V(x)\right) + \sum_{j=1}^{N\Omega}\left(-\frac{\hbar^2}{2}\Delta_{R_j} + \frac{1}{2}\omega_j^2\,|R_j|^2\right) \\
&\quad + \frac{1}{\sqrt{N}}\left(\sum_{j=1}^{N\Omega} C_j R_j\right)\cdot x
\end{aligned}
\tag{1.1}
$$

The first term of (1.) represents the Hamiltonian of the test-particle with mass M where $x \in \mathbb{R}^d$ denotes the test-particle position in dimension d. The abstract reservoir here is a set of finitely many (say $N\Omega$, which is assumed to be integer) independent oscillators written in normal variables $R_j \in \mathbb{R}^d$, having frequencies $\omega_j \in [0, \Omega]$ and masses $m = 1$. Here Ω is the maximum frequency of the oscillators and N is the number of oscillators per unit frequency. The typical case is the uniform frequency distribution: $\omega_j = j/N$ on $[0, \Omega]$. The coupling is linear in x and the R_j's, with coupling coefficients given by the C_j's. The normalization factor $N^{-1/2}$ simply stems from the central limit theorem, since, roughly speaking, the variables R_j's become independent random variables with vanishing expectation in the thermodynamic limit $N \to \infty$. The operator H acts on the Hilbert space $L^2_x(\mathbb{R}^d) \otimes (\bigotimes_{j=1}^{N\Omega} L^2_{R_j}(\mathbb{R}^d))$. The authors of ref. 5 consider only $d = 1$ for simplicity, as we shall do as well, but the method extends to any dimension. A detailed exposition of this model is given in Chapter 4 of ref. 16 or in ref. 53.

Caldeira–Leggett assume that the reservoir is initially in thermal equilibrium at inverse temperature β, i.e., the initial density matrix of the system $A + R$ is given by,

$$\rho^0 = \rho^0_A \otimes \exp(-\beta H_R) \tag{1.2}$$

where ρ^0_A is the initial state of the test-particle. Finally, they choose the coupling coefficients,

$$C_j := \lambda\omega_j \tag{1.3}$$

with some $\lambda > 0$.

Remarks. (i) Instead of uniformly spaced oscillator frequencies $\omega_j = j/N$, it is sufficient to assume that the frequency distribution $\varrho_N(\omega)\, d\omega = (1/N) \sum_{j=1}^{N\Omega} \delta(\omega - \omega_j)\, d\omega$ tends weakly, in the thermodynamic limit ($N \to \infty$), to a uniform distribution $\varrho(\omega)\, d\omega$ on $[0, \Omega]$ with density, say, c, i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N\Omega} h(\omega_j) = \lim_{N\to\infty} \int_0^{\Omega} h(\omega)\, \varrho_N(\omega)\, d\omega = c \int_0^{\Omega} h(\omega)\, d\omega, \quad \forall h \in C[0, \Omega] \tag{1.4}$$

with $\varrho(\omega)$ being c times the characteristic function of $[0, \Omega]$. Without loss of generality $c = 1$ can be assumed because changing c to 1 is equivalent to changing $\lambda \to \sqrt{c}\, \lambda$.

(ii) In fact, the physically relevant quantity is the spectral density of the bath, i.e., the measure

$$J_N(\omega)\, d\omega = \frac{C^2(\omega)}{\omega}\, \varrho_N(\omega)\, d\omega = \frac{1}{N} \sum_{j=1}^{N\Omega} \frac{C_j^2}{\omega_j}\, \delta(\omega - \omega_j) \tag{1.5}$$

with $C_j = C(\omega_j)$ (see (3.23) in ref. 5, apart from constants). In the case of ref. 5, $C(\omega) = \lambda\omega$ and $J_N(\omega)\, d\omega$ converges to the measure $J(\omega)\, d\omega = \lambda^2 \omega \cdot 1(\omega \leqslant \Omega)\, d\omega$ in the limit $N \to \infty$ (here $1(\cdot)$ is the characteristic function). The original model can be considered for any spectral density with arbitrary cutoff (e.g., with the standard Drude cutoff, [16, Section 4.2.3]), but our analysis shows that the assumption $J(\omega) = (const.)\, \omega$ is used for the Caldeira–Leggett derivation in an essential way. However, in Section 6 we present a model where this assumption is not needed to derive a modified Fokker–Planck equation. For a different model in Section 7 we show that the diffusion mechanism is robust; derivation of the Laplacian term in the Fokker–Planck equation does not require uniform frequency distribution. However, in that model the friction term would be time-delayed if ϱ were not uniform.

(iii) We consider a bath of finitely many oscillators, but we will take the thermodynamic limit $N \to \infty$ before any other limit. It is possible to construct the time evolution of the limiting Hamiltonian directly but we prefer to keep the presentation on the most elementary level. For the same reason, we avoid the second quantized formalism. For the phenomena discussed here, there is no need to define a Hamiltonian with infinitely many degrees of freedom and the corresponding Hilbert space; the $N \to \infty$ limit can be taken after the heat bath variables are integrated out as we keep all estimates uniform in N.

(iv) We chose N to denote the number of oscillators per unit frequency instead of the total number of oscillators. Since $N \to \infty$ limit will be taken first, mathematically it is equivalent to letting the total number of oscillators go to infinity. However, in case of the only physical model discussed here (in Section 2), this choice of N will have a physical meaning: it will be the size of the harmonic crystal measured on the lengthscale of the confining potential.

(v) The total potential in (1.1) may be negative, in particular H_{CL} may be unbounded from below as $N \to \infty$. In several related models (see ref. 16) a term

$$\frac{x^2}{2} \cdot \frac{1}{N} \sum_{j=1}^{N\Omega} \frac{C_j^2}{\omega_j^2}$$

is added to the Hamiltonian (1.1) to "complete the square" of the potential in the interaction term and in the bath oscillators. With our choice of C_j this term is $\lambda^2 \Omega(x^2/2)$. Similarly to equation (3.1) in ref. 5, here we prefer not to add this term explicitly to the Hamiltonian, rather we will see that the effective potential acting on the test-particle will be $V_{\text{eff}}(x) = V(x) - \lambda^2 \Omega(x^2/2)$, where the quantity $\lambda^2 \Omega$ is called the frequency shift. This approach is analogous to the procedure followed in Section 3 of ref. 5, see especially equation (3.39). With this choice, the model becomes translation invariant for $V_{\text{eff}} = 0$.

Now the main steps of ref. 5 are the following:

- *First*, using that $H_I + H_R$ is quadratic and relying on Feynman path integrals, Caldeira and Leggett explicitly compute the evolution of the test-particle after tracing out the reservoir variables. The evolution equation of the test-particle involves a diffusive forcing term and a memory term (friction), the latter being non-local in time (see (3.2) below, as well as (4.14)). These terms translate the effect of the evolution of the reservoir on the test-particle. It is very standard in this context that integrating out the reservoir variables gives rise to a non-Markovian evolution for the test-particle, despite that the evolution of the full system is Markovian.

- *Second*, they perform the thermodynamical limit where the number of oscillators (per unit frequency) in (1.1) becomes infinite ($N \to \infty$).

- *Third*, they perform the limit $\Omega \to \infty$, i.e., the frequency range becomes infinite (removing ultraviolet cutoff), let the inverse temperature β go to zero and they perform the semiclassical limit $\hbar \to 0$.

The limits $\Omega \to \infty$ and $\beta \to 0$ allow them to eliminate all the non-Markovian effects. Finally, Caldeira and Leggett state a semiclassical Fokker–Planck equation

$$\partial_t w + v \cdot \nabla_x w - \nabla_x V_{\text{eff}}(x) \cdot \nabla_v w = \gamma \nabla_v (vw) + \sigma \Delta_v w \tag{1.6}$$

for the particle's phase space density $w = w(t, x, v)$, as a result of their asymptotic procedures. The friction coefficient γ is given as $\gamma = \sigma\beta/M$, which is the well-known Einstein's relation between friction, diffusivity and inverse temperature.

As stated above, the last limit Caldeira and Leggett perform to obtain (1.6) is the semiclassical one: $\hbar \to 0$. More importantly, before the semi-classical limit they obtain the quantum Fokker–Planck equation

$$\partial_t w^\hbar + v \cdot \nabla_x w^\hbar - \theta_\hbar[V_{\text{eff}}]\, w^\hbar = \gamma \nabla_v (v w^\hbar) + \sigma \Delta_v w^\hbar \tag{1.7}$$

for the particle's Wigner function $w^\hbar$. Here $\theta_\hbar[V_{\text{eff}}]$ is the pseudo-differential operator (whose symbol depends on the effective potential V_{eff}) describing quantum convection by V_{eff} (see, e.g., ref. 45 for details). Speaking more precisely, Caldeira and Leggett obtain a Markovian evolution equation for the test-particle's density operator whose Wigner transform $w^\hbar$ satisfies (1.7) (cf. e.g., (5.10) in ref. 5).

Note that $w^\hbar = w^\hbar(x, v, t)$ is not pointwise positive. Its weak limit $w = \lim_{\hbar \to 0} w^\hbar$, which solves (1.6), is however pointwise positive, but does not correspond to a quantum evolution (i.e., it is a semiclassical phase space distribution, as said before).

The equation (1.7) is also known under the name of "Quantum Brownian motion," or "Quantum Langevin equation," and received a large interest in the context of interaction between light and matter (see, e.g., ref. 11).

We mention that the idea of formally deriving Fokker–Planck-like equations from a reservoir of oscillators with linear coupling has been exploited by many authors, e.g., refs. 6, 17, 18, 12, 34, and 52 (see also ref. 19 for comments on this equation and the relationship with questions of decoherence). These authors use similar scalings as ref. 5. In particular, in refs. 17, 18, 52, and 34, corrections to (1.6), (1.7) are derived when the temperature is large but finite, and these equations involve both a diffusive term in velocity and friction terms in space and velocity.

Mathematically rigorous work on quantum heat baths is slightly less abundant. The classical paper[27] considers a special non nearest neighbor interaction, so that the evolution of the oscillator chain becomes exactly Markovian after the thermodynamic and semiclassical limits. The result is a quantum Langevin equation and it is very close in spirit to ref. 5. This was probably the first example of a quantum stochastic process derived from a Hamiltonian model. More general constructions are found in ref. 41. A laser model consisting of a chain of two level atoms coupled to a radiation field is rigorously discussed by Hepp and Lieb (see the review paper[32] and references therein). It is shown that the true quantum system converges to the classical laser equations as the number of atoms goes to infinity. A rigorous operator-algebraic approach to quantum heat baths is given in ref. 13, and a path-integral approach is found in ref. 9. A similar model has also been used in the program of Jakšić and Pillet to study thermal relaxation with spectral methods (see ref. 35 and references therein). Recently an analogous system with an extra white-noise is studied in ref. 28. Under different scalings Arai derives ballistic behaviour for the test-particle.[2] In a different context and with different scaling assumptions than ref. 5 and others, but still with the assumption of linear coupling, we also mention ref. 11.

A general framework for weak coupling limits and on obtaining Markovian evolution equations is discussed in a sequence of papers by Davies. Our results in Sections 6 and 7 are analogues of the abstract statements of refs. 14 and 15 for the Caldeira–Leggett Hamiltonians. Our approach is however carried out in the more intuitive Wigner formalism and the limiting Fokker–Planck equations are stated explicitly. They are very illuminating from a kinetic (=phase space) point of view. This is actually the main advantage of the Caldeira–Leggett approach compared with more abstract setups. Note that the abstract results of refs. 14 and 15 do not easily translate into explicit equations in general and kinetic theory in particular.

The key assumption in all these papers is that the test-particle is linearly coupled to the infinite bath of harmonic oscillators, which gives rise to Gaussian computations, and many quantities of interest become explicitly computable. This certainly explains at least part of the interest that these kinds of models have received.

The paper by Caldeira and Leggett raises several questions which have to be addressed. The most serious is that the limiting equation (1.7) is not of Lindblad form (see refs. 1, 18, and 43), which is a generic condition for quantum systems to preserve the complete positivity of the density operator along the evolution. Recall that the true quantum evolution preserves this property. This shortcoming is closely related to the fact, that the equation itself contains β (as the ratio of γ and σ), while $\beta \to 0$ limit was actually used along its derivation. This is not just a mathematical inconsistency. Either the friction term should be negligible compared to the diffusion term in (1.7) if the $\beta \to 0$ limit is really taken; or there should be an extra term in the equation if β is thought of as a small but nonzero number. In the latter case this extra term should restore the Lindblad form of the equation, and it is not clear why this term could be considered negligible compared to the friction.

The confusion probably comes from the unspecified order of limits, which is the second important question and the paper[5] is admittedly vague about it (see comments after (3.33) in ref. 5). In fact, in several cases ref. 5 uses "asymptotic regimes" without taking rigorous limits. The Caldeira–Leggett system relaxes to equilibrium under very mild conditions (see, e.g., ref. 14) without any further limits (apart from $N \to \infty$). However, the precise equation which governs this relaxation depends on the physical parameters of the system. In particular, only in some limiting regimes it is true that the limiting equation is a differential equation (i.e., time-delayed memory terms vanish). Furthermore, to obtain a Fokker–Planck type equation, especially a Laplacian term ($\varDelta_v$), requires further restrictions which are implicitly assumed in various steps of the Caldeira–Leggett derivation. We will demonstrate in particular, that the $\varDelta_v$ term in (1.7) is due to the special choice of the coupling constants $C_k \sim \omega_k$ (or, equivalently, to $J(\omega) \sim \omega$) *and* to the fact that the cutoff frequency $\varOmega$ goes to infinity. In physical systems finite $\varOmega$ is more realistic, but then the resulting equation contains a modified (cutoff) Laplacian, and the system will not be described by a diffusive equation for short times. Although apparently Caldeira–Leggett are not interested in short times (see their remark below (3.35) in ref. 5) they do not formulate this concept rigorously. The scaling limit, we introduce in Sections 6 and 7 will be the precise mathematical tool for this.

Finally, from mathematical point of view, it is desirable to eliminate the nonrigorous steps in the original derivation; especially since the order of limits actually does influence the form of the limiting equation. In addition, the systematic use of the Feynman path integral should be avoided in a rigorous proof, since it is a mathematically undefined object.

We should emphasize that we do not intend to give a full list of equations arising from various regimes of the parameters; and we do not plan to discuss which actual physical systems fall into these regimes. Our purpose is merely to determine the precise conditions and limits which lead *exactly* to a Fokker–Planck equation for the Wigner function (especially with $\varDelta_v$ term). These conditions turn out to be quite restrictive, which does not contradict the fact that the Caldeira–Leggett approach has been used quite extensively and successively in models with phenomenological friction and diffusion mechanisms. There are many different equations which behave similarly to the Fokker–Planck equation, especially if only certain space and time regimes are considered. In fact, we also present two limiting regimes, different from the one implicitly used in ref. 5, which lead to modified Fokker–Planck equations and which use less restrictive assumptions. Their diffusion mechanism does not require uniform frequency distribution (see Remark 1 after Theorem 6.1 and Remark 3 after Theorem 7.1). Moreover, the model in Section 7 does not require high temperature.

The present paper has five parts:

(a) In Section 2 we present a concrete physical model which, in certain approximation, leads to the Caldeira–Leggett Hamiltonian (1.1). This section is for illustration and is independent of the rest of the paper.

(b) In Section 3 we explain that the origin of the diffusive Δ_v term from the original Caldeira–Leggett model is the $\Omega \to \infty$ limit. Then we explain how to modify the model to obtain diffusion via a more realistic mechanism using scaling limit. We also explain how these derivations are related to other derivations of the Fokker–Planck equation via the Boltzmann equation.

(c) In Section 5, we present a rigorous mathematical convergence result for the model introduced in ref. 5. Our approach is very elementary and physically transparent.

(d) In Section 6, we show that one can also recover a diffusive non-kinetic behaviour (frictionless heat equation) from the Caldeira–Leggett Hamiltonian using scaling limit and without assuming infinite frequency range and uniform frequency distribution.

(e) In Section 7, under a different scaling limit, we derive a Fokker–Planck equation with friction but without convective terms. The temperature is finite. Einstein relation is valid in a modified form which takes into account the ground state quantum fluctuations of the heat bath. The diffusion mechanism is independent of the uniformity of the frequency distribution, but the friction term becomes local in time only in this case.

Our main results are Theorem 5.1, 6.1, and 7.1.

Remark. The equation derived in Section 5 is of Lindblad form (see ref. 1). Since there is no rescaling in the variables, one can reconstruct the quantum (restricted) density matrix from the Wigner distribution at every time $t > 0$, hence the equation must preserve the positivity of the corresponding density matrix. The Wigner distribution itself is typically not positive. On the other hand, the heat equations in Sections 6 and 7 are positivity preserving equations in pointwise sense. After rescaling the space-velocity variables (Section 6), the weak limit of the Wigner distribution is a nonnegative phase space density, hence the equation must preserve this property. The time dependent quantum states (density matrices) cannot be reconstructed, but the heat equation determines their rescaled weak limits at any time.

2. ELECTRON IN A HARMONIC IONIC LATTICE

One of the physical situations described by the Caldeira–Leggett Hamiltonian is a single localized electron interacting with phonons. For simplicity, we consider only the one dimensional situation (so called Rubin model, see [16, Section 4.2.4]).

The electron with mass M is subject to a confining potential $V(x)$ and its Hamiltonian is $H_A = -(\hbar^2/2M)\,\Delta_x + V(x)$. We consider units, where $\hbar = M = 1$.

The phonons are generated by a periodic chain of ions, sitting at the points of $\Lambda = \{(j/\Omega) : j = 0, 1, 2,..., N\Omega\} \subset T_N$ where the points 0 and N are identified. Here T_N is the 1-dimensional torus of length N. Let $\Lambda^* = \{(j/N) : j = 0, 1, 2,..., N\Omega\} \subset T_\Omega$ be the dual lattice. Assuming nearest neighbor harmonic coupling, the Hamiltonian of the lattice vibrations is exactly H_R in (1.1) written in normal variables, R_j, which are the Fourier transforms of the ion displacements (see, e.g., ref. 48). After linearization in the phonon variables the interaction of an electron with the crystal lattice is,

$$H_I = \sum_{k \in \Lambda^*} C_k R_k \exp(ik \cdot x) \tag{2.1}$$

where C_k is the kth Fourier component of the electron-photon interaction, which comes from a two-body interaction between the electron and the ions.

The essential point in (2.1) is that this interaction is non-linear in x. One can reach linear coupling by assuming that the quantity $k \cdot x$ in (2.1) remains small during the full evolution of the system, and linearize the exponential accordingly. This means that the wavelength $(= O(|\text{wavevector}|^{-1}) = O(|k|^{-1}))$ of the crystal oscillation should be bigger than the displacement of the particle (x) during its full evolution. Furthermore, in the original Caldeira–Leggett model (as well as in Section 5.3) the ultraviolet cutoff was removed $(\Omega \to \infty)$ in order to obtain diffusion (see Section 3.1). Therefore, we are led to assume big frequencies together with big wavelengths, whereas their product, the sound speed, is a bounded physical constant.

On the level of the Hamiltonian, notice that if C_k were frequency independent (equivalently, $J(\omega) \sim \omega^{-1}$) then $\sum_{k \in \Lambda^*} R_k$, to which the particle coordinate is coupled (1.1), is just the displacement of the ion at the origin as the normal modes are the Fourier transforms of the displacement vectors. In other words, the test-particle is assumed to remain in the vicinity of the origin, and it is assumed to interact with only one single ion of the crystal lattice for all its dynamics (see, e.g., ref. 13). On the other hand, if we wish to derive a diffusive equation for the electron, then for large values of time it is expected to move away from the origin. Even if the diffusion appears only in the velocity (see (1.7)), the large velocity implies large fluctuation in the configuration variable as well.

Coupling depending linearly on the frequency, $C_j \sim \omega_j$, considered in ref. 5, corresponds to $J(\omega) \sim \omega$. Theoretically, it can be obtained from a three-dimensional phonon model with radial coupling. In this case R_j is the sum of all modes R_k with the same frequency ω_j, where k runs through the dual of the three-dimensional lattice Λ. However, we should remark that the Ohmic law $J(\omega) \sim \omega$ breaks down for large frequencies in real systems.

In summary, the linear model effectively involves an implicit mean-field assumption by requiring that the test-particle is coupled to the same mode for all its evolution, which seems incompatible with the finite sound speed of the metals along with the removed UV cutoff. This leaves doubts on the applicability of the linear coupling assumption for diffusion models for electron propagation in an ionic lattice (see also ref. 2 for a brief criticism of this assumption). For electrons coupled to photons (Section 4 in ref. 5) this assumption is more realistic and indeed it is widely used in electromagnetic radiation theory (dipole approximation, see ref. 11).

However, this model is more realistic if $\Omega \to \infty$ is not required, and this is the case for the model discussed in Section 7. Here the electron is subject to a confining potential $V(x) = x^2/2$ and is performing a fast harmonic oscillation. Moreover, it is subject to a weak coupling to the phonons, which slowly modify the phase space support of the fast oscillation. It is this slow motion which is described by a Fokker–Planck equation with friction, after a time rescaling. The electron remains confined in the vicinity of a single ion, hence the linear approximation is more reasonable. Since only the modes near the resonant frequencies are used effectively, the exact form of the spectral bath density $J(\omega)$ is irrelevant for the diffusive mechanism.

3. SOURCE OF DIFFUSION IN VARIOUS KINETIC MODELS

In order to explain the origin of diffusion (Δ_v) in ref. 5, we have to analyze the effects of the limits introduced there. To avoid Feynman path integrals, we will use the characteristics in our proof (see also [16, Section 4.2.2]). Below we present the basic idea of ref. 5 in this language.

We take the Hamiltonian as in ref. 5 (see (1.1)) with $M = 1$ and specify the choice $V(x) = \frac{1}{2}x^2$ (harmonic oscillator), in the spirit of refs. 12, 2, 34, 52, and 11. We use the fact that, for Gaussian Hamiltonians, the evolution equation for the Wigner transform of the density matrix is a first order linear partial differential equation,[54, 44, 30] which can be solved exactly by the method of characteristics (see also ref. 52 for a similar observation).

In the quadratic case, we can scale $\hbar$ out of the equation (1.1). Let

$$H := \frac{1}{2}(-\Delta_x + x^2) + \frac{1}{2}\sum_{j=1}^{N\Omega}(-\Delta_{R_j} + \omega_j^2 R_j^2) + \frac{1}{\sqrt{N}}\left(\sum_{j=1}^{N\Omega} C_j R_j\right)\cdot x \tag{3.1}$$

then $\exp(-it\hbar^{-1}H_{CL})$ and $\exp(-itH)$ are unitarily equivalent under the rescaling of variables $x \to x\hbar^{-1/2}$, $R_j \to R_j\hbar^{-1/2}$, or in other words, we choose units where $\hbar = 1$, $M = 1$.

If $V_{\text{eff}}(x)$ is not quadratic, then it gives rise to a genuine pseudodifferential operator in the Wigner equation and $\hbar$ cannot be scaled out. In the semiclassical limit ($\hbar \to 0$) this term converges to the differential operator $\nabla_x V_{\text{eff}} \cdot \nabla_v w$ in (1.6). This fact is well-known for general semiclassical Wigner equations.[44, 45, 31, 46] We will not prove Theorem 5.1 for a general potential because our main goal is to find the origin of diffusivity which is independent of the confining potential. We restrict ourselves to the most convenient quadratic case.

We also present two different scaling limits starting from (3.1) which allows one to follow the dynamics up to long times. However, we believe that not just our result on the original Caldeira–Leggett model (in Section 5) can be extended to include general potential, but also the resonance effect in Sections 6 and 7. Due to the lack of explicit solutions, this requires extra analysis which we leave to further works.

3.1. Diffusion in the Original Model

After integrating out the reservoir variables in the equations for the characteristics, it eventually reduces to the following ODE for the particle's position variable $X(t)$ (see (4.14) for the exact result),

$$X''(t) + X(t) = f(t) + \lambda^2 \int_0^t S(t-s)\, X(s)\, ds \tag{3.2}$$

Here λ is as in (1.3), S is an explicit function corresponding to the memory effects, and the forcing term f is,

$$f(t) = -\frac{\lambda}{\sqrt{N}} \sum_{j=1}^{N\Omega} \omega_j \left[R_j \cos \omega_j t + P_j \frac{\sin \omega_j t}{\omega_j} \right] \tag{3.3}$$

where R_j, P_j are the initial position and momentum variables of the oscillators. Let $R_j^* := \sqrt{2\beta}\, \omega_j R_j$ and $P_j^* := \sqrt{2\beta}\, P_j$ be their rescaled versions.

In the high temperature limit these become standard Gaussian variables since the classical Gibbs distribution is given by,

$$\prod_j e^{-\beta(P_j^2+\omega_j^2 R_j^2)} = \prod_j e^{-1/2[(P_j^*)^2+(R_j^*)^2]}$$

and at high temperature the quantum Gibbs distribution converges to the classical one (for the precise formulas, see (4.15)–(4.17)). Hence the choice (1.3) for C_j gives that,

$$f(t) = -\frac{\lambda}{\sqrt{2\beta}} \sum_{j=1}^{N\Omega} \left[\frac{R_j^*}{\sqrt{N}} \cos(\omega_j t) + \frac{P_j^*}{\sqrt{N}} \sin(\omega_j t) \right] \tag{3.4}$$

and as $\beta \to 0$, R_j^*, P_j^* approach to standard Gaussians.

After integration by parts in the memory term in (3.2) we obtain (see (4.36))

$$X''(t) + X(t) = f(t) + \lambda^2 \Omega X(t) - (M \star X')(t) - xM(t) \tag{3.5}$$

where M is an approximate Dirac delta function $M(t) \sim \lambda^2 \delta_0(t)$ in the limit $\Omega \to \infty$. Here $\star$ stands for convolution. The term $\lambda^2\Omega$ is the frequency shift of the test-particle oscillator. The friction term $M \star X'$ has a main Markovian part $\lambda^2 X'$ and a non-Markovian part which is negligible as $\Omega \to \infty$.

The effect of the limits introduced in ref. 5 are as follows

- The high temperature limit ($\beta \to 0$) plays two roles. First, it makes the rescaled initial data R_j^*, P_j^* standard Gaussians. Second, it forces the full friction term to be negligible compared to the forcing term.

- In the thermodynamic limit ($N \to \infty$) the sum in (3.4) becomes the sum of the real and imaginary parts of the truncated complex white noise,

$$dW^{(\Omega)}(t) := \int_0^{\Omega} e^{i\omega t} g(d\omega)$$

where $g(d\omega)$'s are independent centered Gaussian random variables with variance $\mathbf{E}[g(d\omega)^2] = d\omega$ (for precise definition see Section 4.4).

- Removing the ultraviolet cutoff ($\Omega \to \infty$) gives the (complex) white noise,

$$dW(t) = \int_0^{\infty} e^{i\omega t} g(d\omega) \tag{3.6}$$

for the forcing term. To prevent instability ($\lambda^2\Omega > 1$), we have to take the simultaneous limit $\lambda \to 0$, $\Omega \to \infty$ which may lead to a nonzero constant phase shift $\lambda^2\Omega$.

Our main concern is to identify the origin of the Δ_v (diffusion) term, which will come from the forcing term (see (3.4)). Hence this term should not vanish in the limit, which indicates that $\beta \to 0$ and $\lambda \to 0$ limits must be related:

$$\lambda = \lambda_0 \beta^{1/2} \qquad (\lambda_0 \text{ fixed}) \tag{3.7}$$

In summary, the solution $X(t)$ to (3.2) converges to the solution of a pure harmonic oscillator with a white noise forcing, i.e., $\theta X(t) + \sigma X'(t) \sim (\eta \star dW)(t)$, where $\eta(s) = \theta \sin s + \sigma \cos s$ is the harmonic oscillator trajectory (with initial condition $\eta(0) = \sigma$, $\eta'(0) = \theta$). In particular the mean square displacement (both in space and velocity)

$$\mathbf{E}\, |\theta X(t) + \sigma X'(t)|^2 \sim \mathbf{E}\, |(\eta \star dW^{(\Omega)})(t)|^2 = \int_0^{\Omega} \left| \int_0^t \eta(t-s)\, e^{-i\omega s}\, ds \right|^2 d\omega \tag{3.8}$$

behaves *quadratically* in t for small t for every finite Ω, hence it is *not* diffusive for short times.

The diffusive behavior (linear mean square displacement) is regained only *after* the $\Omega \to \infty$ limit or after long times.

We emphasize that, from this point of view, the v-Laplacian in the CL model immediately stems from the particular asymptotic distribution of the frequencies (uniform from zero to infinity) in the forcing term. In other terms this model demonstrates diffusion in a setup where a plain diffusive forcing mechanism was essentially put in by hand. Diffusion appears already in very short time scales as a result of high frequency oscillators. This means that there is a shorter, unexplored time scale on which most of the oscillators live, hence the initial Hamiltonian with the Caldeira–Leggett limits should not be considered microscopic, rather mesoscopic. This problem is especially transparent if the heat bath is provided by phonons (crystal lattice vibrations) which have an physical ultraviolet cutoff (lattice spacing). In other words, for systems with UV cutoff and without time rescaling, Δ_v is not the correct diffusion operator.

In contrast to this diffusive mechanism, the source of the diffusion in more realistic models dealing with a moving test-particle interacting with many degrees of freedom is the *scaling limit*, especially time rescaling. This means that in these models the full frequency spectrum of the diffusion is collected over a long time from the cumulative effects of interactions with bounded frequency, and the diffusive behaviour is visible only on a much larger time (and sometimes space) scale than that of the microscopic interaction (collision) mechanism. This makes a key difference between the present model and other works dealing, for instance, with collisional models as scaling limits of microscopic dynamics, i.e., macroscopic long time behaviour of Schrödinger equations (see, e.g., refs. 50, 51, 39, 33, 21–24, 46, 47, 7, 8, and 37 or also ref. 4). We remedy this drawback of the CL scaling in Sections 6 and 7, as we indicate now.

3.2. Diffusion from Resonances in the Scaling Limit

In Section 6, we show that one can also recover a diffusive non-kinetic behaviour from the Caldeira–Leggett Hamiltonian under a more realistic space-time scaling limit. Namely, for a *fixed* cutoff in frequency Ω, and after the high-temperature limit, we consider the resulting dynamics for the test-particle for large time $t \sim \alpha^{-2}$ and large space and velocity variables x, $v \sim \alpha^{-1}$. Here $\alpha \to 0$ is a scaling parameter and we define $X = \alpha x$, $V = \alpha v$, $T = \alpha^2 t$ to be the macroscopic (or rescaled) position, velocity and time variables. We prove that the phase space density is subject to a heat equation both in the (rescaled) velocity and position variables. In particular, the energy of the test-particle increases up to α^{-2} due to the resonances with bath particles of high energy (but bounded frequency). Recall that the temperature of the heat bath is $\beta^{-1} \to \infty$, hence bath particles can have large energy even with bounded frequency.

In this case the diffusion indeed comes from the cumulative effect of bounded frequency interactions via a change of scale. This is in fact a high energy diffusion in phase space; the test-particle is heated up. The forcing frequency distribution can be quite arbitrary, the only condition is that it has to carry energy at the resonant frequency. The diffusion comes from a pure resonance effect, and this seems to be a more universal physical feature in this context (see ref. 11). However, the high temperature limit is still essential in this derivation.

In Section 7, we keep the temperature fixed and we rescale only time, $t = T\delta^{-1}$ (where $\delta \to 0$ plays to role of α^2 above), space and velocity remain unscaled. The reason is that the bath temperature is finite, hence the typical energy ("temperature") of the test-particle remains finite as well. Since the particle Hamiltonian is confining (energy level sets are compact in phase space), the particle remains effectively localized. As a result we get a small scale diffusion in phase space with friction, after integrating out the fast circular motion. Again the diffusion comes from resonance and is developed over a long time period, and the contributing bath frequencies are bounded.

One of the important feature of these models is that the derivation is quite insensitive to the actual form of the spectral density $J(\omega)$ (1.5); the only relevant quantity is its value at the resonant frequency.

3.3. Comparison of the Three Models

The main goal of our investigation is to derive diffusion, i.e., Δ_v term in the limiting equation. The time dependence of the mean square displacement of the characteristics (3.8) is quadratic for small time (unless $\Omega \to \infty$) and is linear for large time. To see diffusion on *all* times considered, there are two alternatives: either we take $\Omega \to \infty$ or we rescale time.

(I) If $\Omega \to \infty$, then the coupling λ must go to zero to keep the frequency shift $\lambda^2\Omega$ finite. Up to a positive time t, the total effect of the friction term is of order $\lambda^2 t$, while the diffusive (forcing) term is roughly of order $\lambda^2 t/\beta$ for larger times, see (5.14), however for short times it is only quadratic in t. Hence for finite times $\lambda^2 t \to 0$, the friction term vanishes. Moreover, the diffusive term vanishes as well, unless $\beta \to 0$ is chosen such that $\lambda^2 \sim \beta$, i.e., the weak coupling and high temperature limits must be related. The frequency shift is $\lambda^2\Omega$ and its actual size depends on the simultaneous limits $\lambda \to 0$, $\Omega \to \infty$. If $\lambda \to 0$ is taken first, then $\Omega \to \infty$, then the frequency shift vanishes. If $\lambda^2\Omega$ is kept at a positive constant along the limits, then we see a frequency shift. These two cases are described in Theorem 5.1, where frictionless Fokker–Planck equations are derived on the microscopic time scale.

(II) If we consider long times, i.e., $t = \alpha^{-2}T$, $\alpha \to 0$ and T is fixed, then the size of the diffusive term is roughly $\lambda^2\alpha^{-2}T/\beta$ for *all* T. To compensate for the blowup α^{-2}, we can either rescale space and velocity ($x = \alpha^{-1}X$, $v = \alpha^{-1}V$) or we set $\lambda^2 \sim \alpha^2$.

(IIa) If we rescale space and velocity as well, then the friction term has a size $\lambda^2 T$ and the diffusion term is of order $\lambda^2 T/\beta$ (in the new variables). One would like to keep λ and β fixed to see both friction and diffusion. But since the phase shift, $\lambda^2\Omega$, has to be kept finite, it forces keeping Ω finite as well. This is the most realistic physical situation. However, the friction has a non-Markovian part, whose size is $\lambda^2 T$ if Ω is fixed (and it goes to zero only if $\Omega \to \infty$). Hence the limiting equation must have a term which is nonlocal in time. This is the extra term which is missing in (1.7), but its inclusion would not lead to to Fokker–Planck, but to an evolution equation with memory.

To derive a differential equation, the non-Markovian friction part has to be killed. With finite Ω it is possible only if $\lambda \to 0$, and then the full friction is eliminated. In order not to eliminate the diffusive term as well, $\beta \sim \lambda^2$ is necessary. This again leads to the high temperature limit, but now Ω is fixed and the diffusion comes from long-time cumulative resonance effects. The fast oscillator motion on the microscopic time scale has to be integrated out; either in time or by a radial averaging. This is the model in Section 6.

(IIb) If we set $\lambda^2 \sim \alpha^2$ and keep β finite, then we see a finite diffusion on a microscopic space and velocity scale. The friction term $\lambda^2 t$ remains positive and the ratio of the friction to the diffusion is β, which gives Einstein relation. Hence Ω could be kept fixed to see the diffusion mechanism.

However, the non-Markovian part of the memory does not vanish unless $\Omega \to \infty$. The qualitative analysis of Section 7 shows that Ω can grow very slowly (like $|\log \alpha|^7$), i.e., the non-Markovian part of the friction is weak for large times and moderately large Ω. This was probably the heuristic idea of Caldeira and Leggett to neglect this term. However, this effect shows up only after time rescaling; for finite microscopic times t this term is not negligible.

Hence we let $\Omega \to \infty$, and assume that $\lambda^2\Omega$ converges to a fixed number (possibly zero). This number gives the frequency shift. Again, we see that the size of the frequency shift delicately depends on the simultaneous limiting procedure. This is the model of Section 7 (where $\delta := \alpha^2$ is introduced for brevity).

We point out that in models IIa and IIb the origin of the diffusion is the time rescaling. Since the forcing frequencies are kept finite, there is no diffusion on the microscopic scale; it becomes visible only after the large time rescaling. Hence the physically questionnable limits, $\beta \to 0$, $\Omega \to \infty$ have nothing to do with the emergence of the diffusion in these models.

However, at least one of these limits is necessary to arrive at a differential equation instead of an integro-differential equation with time delayed memory term. In model IIa (Section 6) we used $\beta \to 0$ and kept Ω fixed, while in IIb (Section 7) we let $\Omega \to \infty$ and kept β finite.

We always consider nonnegative times $t \geqslant 0$. However, most of our computations are valid for *any* time, except those which are directly responsible for the emergence of the diffusion (Laplacian, or linear mean square displacement). We shall point out these steps. If time were evolved backward, $t < 0$, then the same argument would yield an opposite sign of the Laplacian (so that along the evolution it is regularizing) in the final limiting equations. This is the usual phenomenon of irreversibility of the parabolic equations.

3.4. Derivation of Fokker–Planck Equation via Boltzmann Equation

In the Caldeira–Leggett type models we assumed that the test-particle is localized and is subject to a harmonic heat bath with linear interaction. This usually describes particles trapped in a cavity.

For transport phenomena it is more natural to consider a free test-particle subject to a collision mechanism. In these models the collisions are provided by impurities (Lorenz gas) or by a system of many noninteracting particles (Rayleigh gas or phonon models) and one focuses only on the dynamics of the test-particle. The goal is to derive an equation for the reduced phase space distribution from the Hamiltonian dynamics with many degrees of freedom. A scaling limit is necessary to eliminate the details of the single collisions and to keep only their cumulative long-time effects. The effect of a single collision is weakened. One can introduce a weak coupling parameter $\lambda \to 0$; one can consider a gas at low density $\varrho \to 0$ or, in the Rayleigh gas case, one can let the mass ratio of the gas particle and test-particle m/M go to zero. In all cases the time is rescaled as $t = T\delta^{-1}$. The first scale on which collision effects are visible is $\delta \sim \lambda^2$ (weak coupling or van-Hove limit) or $\delta \sim \varrho$ (low density or Grad limit) and $\delta \sim m/M$ (heavy test-particle limit).

In classical mechanics, the limiting equation is the linear Landau equation (or diffusion on the energy surface) for the van-Hove limit;[38] the linear Boltzmann equation for the low density case;[29, 49, 3] and the Fokker–Planck equation for the heavy test-particle case.[20] The Fokker–Planck equation can be obtained in a two step limit as well: first one obtains a linear Boltzmann equation via a low density limit, then a Fokker–Planck equation from a mass rescaling (for an excellent review, see ref. 50).

In quantum mechanics the limiting equation is the linear Boltzmann equation both in the case of the Lorenz gas (see ref. 21 for the low density case and ref. 22 for the weak coupling case) and in the case of the weakly coupled phonons.[23] In the model of ref. 23 a more realistic nonlinear phonon coupling is considered.

In all cases when the first nontrivial limiting equation is Boltzmann, one needs an extra limiting procedure to derive a diffusive equation. For example if the momentum change in the collisions is small (e.g., the mass ratio m/M is small), then a Taylor expansion in the Boltzmann collision operator gives the Fokker–Planck equation in the first nontrivial order (see ref. 42, for rigorous proof[36]). The smallness of the collisions has to be compensated by an extra time rescaling. However, the two step time rescaling cannot be considered as a fully satisfactory derivation since in the first (Boltzmann) limit correlations are neglected which could become relevant on a longer time scale. The proper (but much harder) procedure is to follow the Hamiltonian dynamics up to the desired (larger) time scale.

We remark that a considerably more difficult collision mechanism is when all particles interact, they are identical, and we are interested in the evolution of the one particle marginal distribution (or density matrix). In this case, the limiting equation is expected to be a nonlinear Boltzmann equation and in classical mechanics it was proven by Lanford.[40] In quantum mechanics the correlation structure is complicated and even the first nontrivial (Boltzmann) time scale is not understood rigorously.

Finally, we compare our model IIb to these free kinetic models with collisions. The closest related model is a free electron subject to a weakly coupled phonon interaction considered in ref. 23, where a (linear) Boltzmann equation was derived. In both models the time scale is the van Hove scale $t \sim \lambda^{-2}$, where λ is the coupling constant. In case of the realistic (nonlinear) electron-phonon coupling in ref. 23, each phonon mode contributes equally to the collision mechanism. In the model IIb the source of the diffusion is resonance which originates merely in the test-particle confinement, however for the rigorous proof we need to use the special form of the linear coupling and test-particle Hamiltonian. Phonons with frequencies away from the base frequency of the test-particle Hamiltonian do not contribute, while phonons near the resonance frequency have a strong long time effect. In particular, it is easy to see that the Duhamel expansion used in ref. 23 diverges for the model IIb, which is also an indication that there is no Boltzmann equation behind the Fokker–Planck equation derived in Section 7.

4. PRELIMINARY RESULTS

4.1. The Wigner Formalism

The density matrix,

$$\rho^{N,\varepsilon} := \rho^{N,\varepsilon}(t, x, y, R, Q) \tag{4.1}$$

which is the solution of

$$i\partial_t \rho^{N,\varepsilon} = [H, \rho^{N,\varepsilon}] \tag{4.2}$$

represents the state of the system "particle + reservoir" at time t with the reservoir variables $R = (R_1,..., R_{N\Omega})$, $Q = (Q_1,..., Q_{N\Omega})$. We index the density matrix by N and the superscript $\varepsilon = (\beta, \Omega, \lambda)$ stands for all the other scaling parameters; recall that β is the inverse temperature, Ω is the frequency range and λ is the coupling strength in the Hamiltonian (3.1).

We take the initial data,

$$\rho_A^0 \otimes e^{-\beta H_R} \tag{4.3}$$

with $\rho_A^0 := \rho_A^{N,\varepsilon}(t=0)$ independent of ε. Here $H_R := \frac{1}{2}\sum_{k=1}^{N\Omega}(-\Delta_{R_k} + \omega_k^2 R_k^2)$ is the reservoir Hamiltonian and $\rho_A^{N,\varepsilon}(t,x,y)$ is the density matrix at time t of the test-particle. It is defined by

$$\rho_A^{N,\varepsilon}(t,x,y) := \int_{\mathbb{R}^{N\Omega}} \rho^{N,\varepsilon}(t,x,y,R,R)\,dR \tag{4.4}$$

with the obvious notation $dR = dR_1 \cdots dR_{N\Omega}$. As usual, we do not distinguish between operators and their kernels in the notation. Following ref. 5, we have to compute,

$$Tr_R(e^{-itH}(\rho_A^0 \otimes e^{-\beta H_R})\, e^{itH}) \tag{4.5}$$

where Tr_R is the partial trace over the reservoir variables. We observe that the Hamiltonian (3.1) is quadratic, so that equation (4.2) can actually be transformed into a first order transport partial differential equation by using the Wigner transform. Indeed, let us define the Wigner transform $w^{N,\varepsilon}(t)$ of $\rho^{N,\varepsilon}(t)$ by

$$\begin{aligned} w^{N,\varepsilon}&(t,x,v,R,P) \\ &:= \int_{\mathbb{R}^{N\Omega+1}} \rho^{N,\varepsilon}\left(t, x+\frac{y}{2}, x-\frac{y}{2}, R+\frac{Q}{2}, R-\frac{Q}{2}\right) \\ &\quad \times \exp\left(-i\left[yv + \sum_{k=1}^{N\Omega} Q_k P_k\right]\right) dy\,dQ \end{aligned} \tag{4.6}$$

Also, let us define the Wigner transform of $\rho_A^{N,\varepsilon}$ by

$$w_A^{N,\varepsilon}(t,x,v) := \int_{\mathbb{R}} \rho_A^{N,\varepsilon}\left(t, x+\frac{y}{2}, x-\frac{y}{2}\right)\exp(-iyv)\,dy \tag{4.7}$$

We have the well-known property,

$$w_A^{N,\varepsilon}(t,x,v) := \int_{\mathbb{R}^{2N\Omega}} w^{N,\varepsilon}(t,x,v,R,P)\,dR\,dP \tag{4.8}$$

and the initial datum for $w^{N,\varepsilon}$ is easily computed from (4.3) and the Mehler kernel,

$$w^{N,\varepsilon}(t=0,x,v,R,P) = w_0(x,v)\,W_0^{N,\varepsilon}(R,P) \tag{4.9}$$

with

$$W_0^{N,\varepsilon}(R,P) := \prod_{k=1}^{N\Omega} \left[4\pi \left(\frac{\cosh(\beta\omega_k)-1}{\cosh(\beta\omega_k)+1} \right)^{1/2} \right.$$
$$\times \exp\left(-\left\{ \frac{\omega_k(\cosh(\beta\omega_k)-1)}{\sinh(\beta\omega_k)} R_k^2 \right\} \right)$$
$$\left. \times \exp\left(-\left\{ \frac{\sinh(\beta\omega_k)}{\omega_k(\cosh(\beta\omega_k)+1)} P_k^2 \right\} \right) \right]$$

Here, $w_0(x, v)$ is the initial datum for the test-particle, i.e., it is the Wigner transform of $\rho_A^0(x, y)$. Here and in the sequel, we shall assume the following regularity for w_0,

$$\hat{w}_0(\xi,\eta) := \int_{\mathbb{R}^2} w_0(x,v) \exp(-i[x\xi + v\eta])\, dx\, dv \in L^1(\mathbb{R}_\xi \times \mathbb{R}_\eta) \tag{4.10}$$

It is well known that, if $\rho^{N,\varepsilon}$ satisfies the Von-Neumann equation (4.2) with Hamiltonian given by (3.1), then its Wigner transform (4.6) satisfies the following partial differential equation,

$$\partial_t w^{N,\varepsilon} + v\,\partial_x w^{N,\varepsilon} - x\,\partial_v w^{N,\varepsilon} + \sum_{k=1}^{N\Omega} (P_k\,\partial_{R_k} w^{N,\varepsilon} - \omega_k^2 R_k\,\partial_{P_k} w^{N,\varepsilon})$$
$$-\frac{\lambda}{\sqrt{N}} \left(\sum_{k=1}^{N\Omega} \omega_k R_k \right) \partial_v w^{N,\varepsilon} - \frac{\lambda}{\sqrt{N}} \left(\sum_{k=1}^{N\Omega} \omega_k x\,\partial_{P_k} w^{N,\varepsilon} \right) = 0 \tag{4.11}$$

As a conclusion we can now rephrase our original problem in the Wigner formalism: following ref. 5, we want to derive a diffusive behaviour for $w_A^{N,\varepsilon}(t)$, the trace of $w^{N,\varepsilon}(t)$, in the thermodynamic limit ($N \to \infty$) and in certain limiting regimes of ε. Here, $w^{N,\varepsilon}$ satisfies (4.11) with initial datum (4.9).

4.2. Solution by Characteristics

Equation (4.11) can easily be solved by the method of characteristics. In fact, for all values of time t, and for all smooth, compactly supported test functions $\phi(x, v)$, we have,

$$\int_{\mathbb{R}^2} w_A^{N,\varepsilon}(t,x,v)\,\bar{\phi}(x,v)\,dx\,dv$$

$$= \int_{\mathbb{R}^{2N\Omega+2}} w(t=0,x,v,R,P)\,\bar{\phi}(X(t),V(t))\,dx\,dv\,dR\,dP$$

$$= \int_{\mathbb{R}^{2N\Omega+6}} \hat{w}_0(\xi,\eta)\,\overline{\hat{\phi}(\theta,\sigma)}\,e^{i(x\xi+v\eta)}$$

$$\times e^{-i(X(t)\,\theta+V(t)\,\sigma)}W_0^{N,\varepsilon}(R,P)\,dx\,dv\,dR\,dP\,d\xi\,d\eta\,d\theta\,d\sigma \tag{4.12}$$

where we have introduced the (forward) characteristics,

$$\begin{aligned} X'(t) &= V(t), \qquad & V'(t) &= -X(t) - \frac{\lambda}{\sqrt{N}}\sum_{k=1}^{N\Omega}\omega_k R_k(t) \\ R_k'(t) &= P_k(t), \qquad & P_k'(t) &= -\omega_k^2 R_k(t) - \frac{\lambda}{\sqrt{N}}\omega_k X(t) \end{aligned} \tag{4.13}$$

with initial data $X(0)=x$, $V(0)=v$, $R_k(0)=R_k$ and $P_k(0)=P_k$. Here we used that the flow (4.13) preserves the Lebesgue measure over $\mathbb{R}^{2(N\Omega+1)}$. For simplicity, we did not index the characteristics by N, ε, but $X(t)$, $V(t)$ in (4.12) depend on N, ε. However, sometimes we will use $X_N(t)$ for special emphasis.

Integrating with respect to $R_k(t)$ in (4.13) and inserting the result in the equation for $X(t)$ gives,

$$\begin{aligned} X''(t) + X(t) = &-\frac{\lambda}{\sqrt{N}}\sum_{k=1}^{N\Omega}\omega_k\left[R_k\cos\omega_k t + P_k\frac{\sin\omega_k t}{\omega_k}\right] \\ &+\frac{\lambda^2}{N}\sum_{k=1}^{N\Omega}\int_0^t \omega_k \sin\omega_k(t-s)\,X(s)\,ds \end{aligned} \tag{4.14}$$

The right-hand-side of (4.14) is of the form "forcing term + memory term" (see also [16, Section 4.2.2]).

In view of (4.9) and (4.12), the partial trace over the oscillators is an integral with respect to a Gaussian distribution in R_k, P_k with (unnormalized) density,

$$\exp\left[-\frac{\omega_k(\cosh\beta\omega_k - 1)}{\sinh\beta\omega_k}R_k^2 - \frac{\sinh\beta\omega_k}{\omega_k(\cosh\beta\omega_k+1)}P_k^2\right] \tag{4.15}$$

Changing variables such that,

$$r_k = \sqrt{\frac{2\omega_k(\cosh\beta\omega_k - 1)}{\sinh\beta\omega_k}}\, R_k, \qquad p_k = \sqrt{\frac{2\sinh\beta\omega_k}{\omega_k(\cosh\beta\omega_k + 1)}}\, P_k \tag{4.16}$$

we obtain (after normalization) the standard Gauss measure,

$$d\mu_N = \prod_{k=1}^{N\Omega} \frac{1}{2\pi}\, e^{-1/2(r_k^2 + p_k^2)}\, dr_k\, dp_k \tag{4.17}$$

i.e., r_k, p_k are independent standard Gaussian variables. The integration with respect to this probability measure will be denoted by $\mathbf{E}_N$.

Using these new variables and integration by parts with respect to s, the equation (4.14) for $X_N(t) = X(t)$ becomes,

$$X_N''(t) + X_N(t) = f_N(t) + \lambda^2 \Omega X_N(t) - (M_N \star X_N')(t) - x M_N(t) \tag{4.18}$$

with

$$f_N(t) := -\frac{\lambda}{\sqrt{N}} \sum_{k=1}^{N\Omega} A_\beta(\omega)[r_k \cos\omega_k t + p_k \sin\omega_k t] \tag{4.19}$$

and

$$M_N(t) := \frac{\lambda^2}{N} \sum_{k=1}^{N\Omega} \cos\omega_k t \tag{4.20}$$

Here we defined,

$$A(\omega) = A_\beta(\omega) := \sqrt{\frac{\omega(\cosh\beta\omega + 1)}{2\sinh\beta\omega}} \tag{4.21}$$

We see that the memory term is split into three parts. The term $\lambda^2\Omega X_N$ induces a frequency shift of the test-particle oscillator, $M_N \star X_N'$ is the friction term and the last inhomogeneous term will be irrelevant. We define

$$a^2 = a_\varepsilon{}^2 := 1 - \lambda^2 \Omega$$

(recall that ε stands for the triple (β, Ω, λ)), and we always assume that a_ε is uniformly separated from zero, i.e., $c_0 \leqslant a_\varepsilon \leqslant 1$ with some constant $c_0 > 0$. We can rewrite (4.18) as

$$X_N''(t) + a^2 X_N(t) = f_N(t) - (M_N \star X_N')(t) - x M_N(t) \tag{4.22}$$

4.3. The Thermodynamic Limit

We now perform the limit $N \to \infty$. A possible way is to solve (4.14) (iteratively), and compute the limit in the corresponding formulae (see (4.42) later). This rigorously gives the thermodynamic limit but we present an alternative approach which is more illuminating to explain the asymptotic diffusion that we shall recover in Section 5.3. We first need an a priori bound.

Lemma 4.1. Let $X_N(t)$ solve (4.22) with initial conditions $X(0) = x$, $X'(0) = v$, and let

$$F_N(t) := \sup_{s \leqslant t} \mathbf{E}_N \, |X_N(t)| + \sup_{s \leqslant t} \mathbf{E}_N \, |X'_N(t)| \tag{4.23}$$

Then there is a constant $C > 0$ such that

$$F_N(t) \leqslant Ce^{Kt}(|x| + |v| + K\,|x| + \sup_{s \leqslant t} \{se^{-Ks}\} [\lambda^2 \Omega(\beta^{-1} + \Omega)]^{1/2}) \tag{4.24}$$

uniformly in N, where

$$K = K(\lambda, \Omega) := C\lambda^2 \left(1 + \frac{1}{|\Omega - a|}\right) \tag{4.25}$$

and $a^2 = 1 - \lambda^2 \Omega \in (0, 1]$.

Proof. From the fundamental solution of (4.22), one has

$$\begin{aligned} X_N(t) &= x \cos at + va^{-1} \sin at \\ &\quad + \int_0^t a^{-1} \sin a(t-s)[f_N(s) - (M_N \star X'_N)(s) - xM_N(s)] \, ds \\ X'_N(t) &= -xa \sin at + v \cos at \\ &\quad + \int_0^t \cos a(t-s)[f_N(s) - (M_N \star X'_N)(s) - xM_N(s)] \, ds \end{aligned} \tag{4.26}$$

First step. To estimate the memory term in (4.26), we write,

$$\begin{aligned} &\int_0^t \sin[a(t-s)](M_N \star X'_N)(s) \, ds \\ &\quad = (\sin(a\cdot) \star M_N \star X'_N)(t) \\ &\quad = \int_0^t \left(\int_0^s \sin[a(s-u)] \, M_N(u) \, du \right) X'_N(t-s) \, ds \end{aligned} \tag{4.27}$$

An easy calculation shows that the inner integral is bounded by

$$\left|\int_0^s \sin[a(s-u)]\, M_N(u)\, du\right| = |(M_N \star \sin(a\cdot))(s)| \leqslant k\lambda^2 \left(1+\frac{1}{|a-\Omega|}\right) \tag{4.28}$$

with a universal constant k uniformly in N. Indeed, notice that,

$$\lim_{N\to\infty} M_N(s) = \lambda^2 \frac{\sin \Omega s}{s} =: M(s) \tag{4.29}$$

uniformly for $s \in [0, t]$. Moreover $\int_0^s \sin[a(s-u)]\, M(u)\, du$ can be estimated by splitting the integration into two regimes $u \leqslant 1$ and $u \geqslant 1$ (or $u \leqslant s$ regime only if $s \leqslant 1$) and both regimes can be estimated by elementary integration by parts to obtain (4.28).

Hence the expected value of the integral of the memory terms in (4.26) is estimated by,

$$\begin{aligned} &\mathbf{E}_N \left|\int_0^t a^{-1} \sin a(t-s)[-(M_N \star X'_N)(s) - xM_N(s)]\, ds\right| \\ &\quad \leqslant a^{-1} k\lambda^2 \left(1+\frac{1}{|a-\Omega|}\right)\left[|x| + \int_0^t F_N(s)\, ds\right] \end{aligned} \tag{4.30}$$

and similarly for the cosine term in (4.26).

Second step. For the forcing term one computes,

$$\mathbf{E}_N \left|\int_0^t \sin[a(t-s)]\, f_N(s)\, ds\right| \leqslant t \sup_{s\leqslant t} (\mathbf{E}_N\, |f_N(s)|^2)^{1/2} \tag{4.31}$$

We have,

$$\mathbf{E}_N\, |f_N(s)|^2 = \frac{\lambda^2}{N} \sum_{k=1}^{N\Omega} A_\beta^2(\omega) \leqslant \hat{k}\lambda^2 \Omega(\beta^{-1} + \Omega) \tag{4.32}$$

where $\hat{k}$ is again some positive constant, independent of N. Indeed, this sum is an approximating Riemann sum for the integral,

$$\lambda^2 \int_0^\Omega A_\beta^2(\omega)\, d\omega = \lambda^2 \int_0^\Omega \frac{\omega(\cosh \beta\omega + 1)}{2 \sinh \beta\omega}\, d\omega$$

which satisfies the estimate (4.32). Hence we obtain,

$$\mathbf{E}_N[|X_N(t)| + |X'_N(t)|] \leqslant |x| + |v| + k\lambda^2\left(1 + \frac{1}{|a-\Omega|}\right)\left[|x| + \int_0^t F_N(s)\,ds\right] + t[\hat{k}\lambda^2\Omega(\beta^{-1}+\Omega)]^{1/2} \tag{4.33}$$

By a standard Gronwall-type argument we conclude (4.24). ∎

4.4. Digression on Stochastic Integrals

Stochastic integration is integration with respect to a random measure. Once the measure is specified, the integrals are defined as limits of integrals of stepfunctions. We do not develop this notion here, just indicate how it is related to the present problem.

Definition 4.1. The ensemble of random variables $g(A)$, A running over the Borel sets of $\mathbb{R}$, is called *standard Gaussian random measure* if $g(A)$ is a centered real Gaussian random variable for all A and $\mathbf{E}g(A)\,g(B) = |A \cap B|$ where $|\cdot|$ is the Lebesgue measure.

In the thermodynamic limit $N \to \infty$, the forcing term (4.19) converges in an $L^2(d\mu_N)$ sense towards the stochastic integral,

$$f(t) := -\lambda \int_0^{\Omega} A_\beta(\omega)[r(d\omega)\cos\omega t + p(d\omega)\sin\omega t] \tag{4.34}$$

where $r(d\omega)$, $p(d\omega)$ are independent standard Gaussian random measures. The expectation with respect to their joint measure is denoted by $\mathbf{E}$. Clearly $f_N(t)$ is a Riemann sum approximation of $f(t)$ by choosing $r_k := N^{1/2}r([(k-1)/N, k/N])$ and $p_k := N^{1/2}p([(k-1)/N, k/N])$, since their distribution is $d\mu_N$ (see (4.17)). In particular we can realize all f_N's and f on a common probability space. Note that $f(t)$ is formally a white noise (see (3.6)) when the "hyperbolic factor" $A_\beta(\omega)$ is replaced by one and $\Omega = \infty$.

Lemma 4.2. For $1 < \Omega < \infty$ there exist a random function $X(t)$ such that,

$$\lim_{N\to\infty}\left(\sup_{s\leqslant t}\mathbf{E}\,|X_N(s) - X(s)| + \sup_{s\leqslant t}\mathbf{E}\,|X'_N(s) - X'(s)|\right) = 0 \tag{4.35}$$

and $X(t)$ almost surely satisfies the equation,

$$X''(t) + a^2X(t) = f(t) - (M \star X')(t) - xM(t) \tag{4.36}$$

with initial conditions $X(0)=x$, $X'(0)=v$. Moreover,

$$F(t) := \sup_{s \leqslant t} \mathbf{E}\,|X(s)| + \sup_{s \leqslant t} \mathbf{E}\,|X'(s)|$$

satisfies the same estimate as $F_N(t)$ (see (4.24)),

$$F(t) \leqslant Ce^{Kt}(|x| + |v| + K\,|x| + \sup_{s \leqslant t}\{se^{-Ks}\}[\lambda^2 \Omega(\beta^{-1}+\Omega)]^{1/2}) \tag{4.37}$$

Proof. Let us define $X(t)$ by the integral equation,

$$\begin{aligned} X(t) &= x \cos at + va^{-1} \sin at \\ &\quad + \int_0^t a^{-1} \sin[a(t-s)][f(s) - (M \star X')(s) - xM(s)]\, ds \end{aligned} \tag{4.38}$$

Since,

$$\int_0^t \mathbf{E}\,|f(s)|^2\, ds = \lambda^2 \int_0^\Omega \frac{\omega(\cosh \beta\omega + 1)}{2\sinh \beta\omega}\, d\omega < \infty$$

$X(t)$ is well defined almost surely and satisfies (4.36). Moreover, the uniformity of (4.24) in N, and (4.35) shows that $F(t)$ satisfies (4.37). So we are left with proving (4.35).

Let $Z_N(s) := X_N(s) - X(s)$, then it satisfies (from (4.26) and (4.38)),

$$\begin{aligned} Z_N(t) = \int_0^t a^{-1} \sin[a(t-s)][&f_N(s) - f(s) - (M \star Z'_N)(s) \\ &- (M_N - M) \star X'_N(s) - x(M_N - M)(s)]\, ds \end{aligned}$$

and a similar formula holds $Z'_N(t)$. Clearly $Z_N(0) = Z'_N(0) = 0$. Hence, similarly to (4.33),

$$\begin{aligned} &\mathbf{E}(|Z_N(s)| + |Z'_N(s)|) \\ &\quad \leqslant K \int_0^t \tilde{F}_N(s)\, ds + a^{-1} t \sup_{s \leqslant t} (\{|x| + t \sup_{u \leqslant t} \mathbf{E}\,|X'_N(u)|\}\, |M_N(s) - M(s)| \\ &\qquad + \mathbf{E}\,|f_N(s) - f(s)|) \end{aligned}$$

with $\tilde{F}_N(t) = \sup_{s \leqslant t} \mathbf{E}\,|Z_N(s)| + \sup_{s \leqslant t} \mathbf{E}\,|Z'_N(s)|$. We use again a Gronwall argument to obtain (4.35), based upon the control of $\sup_{u \leqslant t} \mathbf{E}\,|X'_N(u)|$ from Lemma 4.1 and the facts that $|M_N(s) - M(s)| \to 0$ (see (4.29)) and $\mathbf{E}\,|f_N(s) - f(s)| \to 0$ uniformly for $s \leqslant t$ as $N \to \infty$.

In order to check $\mathbf{E}\,|f_N(s)-f(s)|\to 0$, we observe that,

$$r_k = N^{1/2} r\left(\left[\frac{k-1}{N}, \frac{k}{N}\right]\right) = N^{1/2}\int \mathbf{1}\left(\omega\in\left[\frac{k-1}{N}, \frac{k}{N}\right]\right) r(d\omega)$$

to obtain,

$$\mathbf{E}\,|f(s)-f_N(s)|^2 = \lambda^2\int_0^{\Omega}\left[A_\beta(\omega)-\sum_{k=1}^{N\Omega} A_\beta(\omega_k)\cdot\mathbf{1}\left(\omega\in\left[\frac{k-1}{N}, \frac{k}{N}\right]\right)\right]^2 d\omega \tag{4.39}$$

which goes to zero as $N\to\infty$, uniformly in $s\leqslant t$. For uniformly spaced frequencies, $\omega_k = k/N$, (4.39) is straightforward. For frequencies satisfying only the uniform density condition (1.4) with $c=1$, first one has to verify that

$$\lim_{N\to\infty}\frac{1}{N}\,\#\left\{k : \left|\omega_k - \frac{k}{N}\right| \geqslant \eta\right\} = 0$$

for any $\eta>0$, and then using the continuity of the function $A_\beta(\omega)$ to conclude the result. ▮

Let us remark that for the present paper there is no need to use stochastic integrals. A reader who is unfamiliar with this concept, can keep the finite sums $\sum_{k=1}^{N\Omega}$ instead of $\int_0^\Omega d\omega$, $f_N(t)$ instead of $f(t)$, and keep on thinking of $\mathbf{E}$ as expectation $\mathbf{E}_N$ with respect to the finite dimensional measure $d\mu_N$. We shall compute various expectations involving $f(t)$. The results are given as an ordinary $\int_0^\Omega(\cdots)\,d\omega$ integral. However, one can keep the finite dimensional approximations $f_N(t)$, and perform the expectations with respect to $d\mu_N$. In this case the expectations involve a finite sum over the frequencies, like $\sum_{k=1}^{N\Omega}(\cdots)$. It is sufficient to take the $N\to\infty$ limit only in this sum, which is a Riemann sum for the integral $\int_0^\Omega(\cdots)\,d\omega$ using (1.4) with $c=1$. However, for notational simplicity we will use the continuous formalism. Note that the thermodynamic limit $N\to\infty$ is always taken before any other limits.

The conclusion of Section 4 is the

Lemma 4.3. Assume (1.4) with $c=1$ and assume (4.10). Let $w_A^{N,\varepsilon}(t)$ be defined as (4.8), while $w^{N,\varepsilon}(t)$ is the solution of (4.11) with initial datum (4.9). Then, in the thermodynamic limit, we have for all $\phi(x,v)\in C_c^\infty(\mathbb{R}^2)$ locally uniformly for $t\in\mathbb{R}$,

$$\lim_{N\to\infty}\int_{\mathbb{R}^2} w_A^{N,\varepsilon}(t,x,v)\,\bar{\phi}(x,v)\,dx\,dv = \int_{\mathbb{R}^2} w_A^{\varepsilon}(t,x,v)\,\bar{\phi}(x,v)\,dx\,dv \tag{4.40}$$

where w_A^ε is defined by

$$\int_{\mathbb{R}^2} w_A^\varepsilon(t, x, v)\, \bar{\phi}(x, v)\, dx\, dv$$

$$= \mathbf{E} \int_{\mathbb{R}^6} \hat{w}_0(\xi, \eta)\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{i(x\xi + v\eta)} e^{-i(X(t)\,\theta + X'(t)\,\sigma)}\, d\xi\, d\eta\, dx\, dv\, d\theta\, d\sigma \tag{4.41}$$

and X satisfies (4.36).

For the proof one only has to observe that the dominated convergence theorem applies and use Lemma 4.2 and (4.12) (recalling that X is actually X_N in that formula). ∎

Remark. As an alternative proof which avoids any reference to probabilistic concepts, we can easily compute the right-hand side of (4.12) directly by performing a finite dimensional Gaussian integration with respect to $d\mu_N$ (again, $X(t)$ is actually $X_N(t)$ in (4.12)). In this case all the integrals $\int_0^{N\Omega} (\cdots)\, d\omega$ are finite sums and the $N \to \infty$ limit is taken only after having performed the $d\mu_N$ integration. We easily find that the right-hand side of (4.12) is equal to,

$$\int_{\mathbb{R}^2} \hat{w}_0(A(t)\,\theta + A'(t)\,\sigma, B(t)\,\theta + B'(t)\,\sigma)\, \overline{\hat{\phi}(\theta, \sigma)}$$

$$\times \exp\left[-\int_0^\Omega \frac{[A_\omega(t)\,\theta + A'_\omega(t)\,\sigma]^2}{2\lambda_\omega}\, d\omega - \int_0^\Omega \frac{[B_\omega(t)\,\theta + B'_\omega(t)\,\sigma]^2}{2\mu_\omega} d\omega \right] d\theta\, d\sigma \tag{4.42}$$

where $\lambda_\omega = [2\omega(\cosh(\beta\omega) - 1)]/[\sinh(\beta\omega)]$, $\mu_\omega = [2\sinh(\beta\omega)]/[\omega(\cosh(\beta\omega) + 1)]$, and,

$$\Psi(t) = \lambda^2 \int_0^\Omega \int_0^t \omega \sin(\omega[t - s]) \sin(s)\, ds\, d\omega$$

$$A(t) = \cos(t) + (\Psi \star A)(t)$$

$$B(t) = \sin(t) + (\Psi \star B)(t)$$

$$A_\omega(t) = -\int_0^t \lambda\omega \cos(\omega s) \sin(t - s)\, ds + (\Psi \star A_\omega)(t)$$

$$B_\omega(t) = -\int_0^t \lambda \sin(\omega s) \sin(t - s)\, ds + (\Psi \star B_\omega)(t)$$

5. THE FOKKER–PLANCK EQUATION FROM THE ORIGINAL CALDEIRA–LEGGETT MODEL

5.1. Evolution Without Friction

In the spirit of ref. 5, we would like to exhibit a scaling where the solution of (4.36) is close to the solution $\tilde{X}(t)$ of the equation without friction term below. The scaling parameters are $\varepsilon = (\beta, \Omega, \lambda)$. The frictionless equation (compare with (4.36)) is,

$$\tilde{X}''(t) + a^2 \tilde{X}(t) = f(t) \qquad \text{with} \quad \tilde{X}(0) = x, \quad \tilde{X}'(0) = v \tag{5.1}$$

recalling that $a^2 = a_\varepsilon{}^2 = 1 - \lambda^2 \Omega \in (0, 1]$.

We need a continuity result ensuring that $X(t)$ and $\tilde{X}(t)$ are close. If $Y(t) = X(t) - \tilde{X}(t)$, then,

$$Y''(t) + a^2 Y(t) = -(M \star X')(t) - xM(t) \tag{5.2}$$

with initial conditions $Y(0) = Y'(0) = 0$. Given the bound (4.37) on $X(t)$ and (4.28) it is trivial to see that,

$$\begin{aligned} &\mathbf{E}(|Y(t)| + |Y'(t)|) \\ &\quad \leqslant Kte^{Kt}\Big(|x| + |v| + K\,|x| + \sup_{s \leqslant t} \{se^{-Ks}\}\, [\lambda^2 \Omega(\beta^{-1} + \Omega)]^{1/2}\Big) \end{aligned} \tag{5.3}$$

where $K = C\lambda^2(1 + (1/|\Omega - a|))$ (see (4.25)). So in particular the solution of (4.36) tends to the solution of (5.1) in a very strong norm if the right-hand side of (5.3) goes to zero. This happens for example for such limiting regimes of $\varepsilon = (\beta, \Omega, \lambda)$ that $\lambda \to 0$ and $\Omega \to \infty$ in such a way that $a^2 = 1 - \lambda^2 \Omega \in (0, 1]$ and $\lambda^2 \beta^{-1/2} \to 0$.

Hence, as soon as one can ensure a small right-hand side in (5.3), we can replace X by $\tilde{X}$ in (4.40)–(4.41) by the Lebesgue theorem, since the x, v, θ, σ integrations range over a bounded domain (ϕ is compactly supported) and we assumed $\hat{w}_0(\xi, \eta) \in L^1$ (see (4.10)). This proves

Lemma 5.1. Let $\tilde{w}^\varepsilon_A$ be defined as,

$$\begin{aligned} &\int_{\mathbb{R}^2} \tilde{w}^\varepsilon_A(t, x, v)\, \bar{\phi}(x, v)\, dx\, dv \\ &\qquad = \mathbf{E} \int_{\mathbb{R}^6} \hat{w}_0(\xi, \eta)\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{i(x\xi + v\eta)} e^{-i(\tilde{X}(t)\,\theta + \tilde{X}'(t)\,\sigma)}\, d\xi\, d\eta\, dx\, dv\, d\theta\, d\sigma \end{aligned} \tag{5.4}$$

analogously to (4.41). Then,

$$\lim_{\varepsilon} \int_{\mathbb{R}^2} \tilde{w}^{\varepsilon}_A(t, x, v)\, \bar{\phi}(x, v)\, dx\, dv = \lim_{\varepsilon} \int_{\mathbb{R}^2} w^{\varepsilon}_A(t, x, v)\, \bar{\phi}(x, v)\, dx\, dv \tag{5.5}$$

for any limit of the parameters $\varepsilon = (\beta, \Omega, \lambda)$ for which the right-hand side of (5.3) goes to zero. ∎

5.2. Computing the Dynamics of the Test-Particle When the Memory Vanishes

In this section we compute $w^{\varepsilon}(t, x, v)$ when X is actually replaced by $\tilde{X}$, the solution of (5.1), in (4.41). We have,

$$\begin{aligned} \tilde{X}(t) &= x \cos at + va^{-1} \sin at + \int_0^t a^{-1} \sin a(t-s)\, f(s)\, ds \\ \tilde{X}'(t) &= -xa \sin at + v \cos at + \int_0^t \cos[a(t-s)]\, f(s)\, ds \end{aligned} \tag{5.6}$$

Hence

$$\begin{aligned} &\int_{\mathbb{R}^2} \tilde{w}^{\varepsilon}_A(t, x, v)\, \overline{\phi(x, v)}\, dx\, dv \\ &\quad = \mathbf{E} \int_{\mathbb{R}^6} \hat{w}_0(\xi, \eta)\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{i(x\xi + v\eta)} e^{-i(\tilde{X}(t)\,\theta + \tilde{X}'(t)\,\sigma)}\, d\xi\, d\eta\, dx\, dv\, d\theta\, d\sigma \\ &\quad = \mathbf{E} \int_{\mathbb{R}^2} \hat{w}_0(\xi_{\theta, \sigma}(t), \eta_{\theta, \sigma}(t))\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{-i \int_0^t (t-s)\, f(s)\, ds}\, d\theta\, d\sigma \end{aligned} \tag{5.7}$$

with

$$\eta_{\theta, \sigma}(t) := \theta a^{-1} \sin at + \sigma \cos at, \qquad \xi_{\theta, \sigma}(t) := \theta \cos at - \sigma a \sin at \tag{5.8}$$

which are, by the way, harmonic oscillator trajectories,

$$\frac{d}{dt} \eta_{\theta, \sigma}(t) = \xi_{\theta, \sigma}(t), \qquad \frac{d}{dt} \xi_{\theta, \sigma}(t) = -a^2 \eta_{\theta, \sigma}(t) \tag{5.9}$$

After performing the expectation in (5.7), we arrive at

Lemma 5.2. With the notations above, we have for any $t \geqslant 0$,

$$\int_{\mathbb{R}^2} \tilde{w}_A^\varepsilon(t, x, v)\, \overline{\phi(x, v)}\, dx\, dv = \int_{\mathbb{R}^2} \hat{w}_0(\xi_{\theta,\sigma}(t), \eta_{\theta,\sigma}(t))\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{-1/2 Q(t)}\, d\theta\, d\sigma \tag{5.10}$$

with

$$Q(t) := Q(t; \theta, \sigma; \beta, a) = \lambda^2 \int_0^\Omega A_\beta{}^2(\omega)\, H(t, \omega)\, d\omega \tag{5.11}$$

$$H(t, \omega) := H(t, \omega; \theta, \sigma; a) = \left| \int_0^t \eta_{\theta,\sigma}(s)\, e^{-i\omega s}\, ds \right|^2 \tag{5.12}$$

The functions $\xi_{\theta,\sigma}$, $\eta_{\theta,\sigma}$ are defined by (5.8). The function $H(t, \omega)$ satisfies the following estimate

$$H(t, \omega) \leqslant 2\gamma^2 \left\{ \left| \frac{e^{it(a-\omega)} - 1}{a - \omega} \right|^2 + \frac{4}{(a+\omega)^2} \right\} \tag{5.13}$$

with $\gamma^2 := \theta^2 + a^2\sigma^2$. Assuming $\Omega > 1$ we also have

$$Q(t) = I\lambda^2 t\gamma^2 \frac{\cosh \beta a + 1}{2a \sinh \beta a} + \lambda^2\gamma^2 B(t) \tag{5.14}$$

with $I := \pi/2$ and with a function B satisfying $B(0) = 0$ and

$$|B(t)| \leqslant C[1 + \beta^{-1}][1 + (\log t)_+][1 + \log \Omega] \tag{5.15}$$

with a universal constant C. Also, we have the estimate:

$$Q(t) = \mathbf{E}(f \star \eta_{\theta,\sigma})^2\,(t) = \mathbf{E}(\theta \widetilde{X}(t) + \sigma \widetilde{X}'(t))^2 + \mathcal{O}[(|x| + |v|)(|\theta| + |\sigma|)] \tag{5.16}$$

Remark 1. Notice that $Q(t)$ grows quadratically in t for small t (since H does so). This means that the test-particle as described by the Wigner distribution w_A^ε has a ballistic behaviour when the memory effects disappear (quadratic growth of the mean squared displacement $\mathbf{E}\widetilde{X}^2(t)$). In the rest of this paper we show that, under several specific scaling limits, one can indeed replace w_A^ε with $\tilde{w}_A^\varepsilon$ (see Lemma 5.1) *and* recover a linear growth for $Q(t)$, i.e., a diffusive behaviour for the test-particle. In particular, this is where the time asymmetric condition $t \geqslant 0$ is used.

Remark 2. Suppose that the frequency distribution $\varrho(\omega)$ (see (1.4)) is not uniform (hence $J(\omega)$ is not linear). By the same calculation, we still obtain (5.10) except that $Q(t)$ is given by $\lambda^2 \int_0^\Omega A_\beta^2(\omega)\, H(t,\omega)\, \varrho(\omega)\, d\omega$. Assuming that $\varrho(\omega)$ is bounded and it is differentiable around the resonant frequency $\omega = a$, we obtain the analogue of (5.14),

$$Q(t) = I\lambda^2 t\gamma^2 \varrho(a) \frac{\cosh \beta a + 1}{2a \sinh \beta a} + \lambda^2\gamma^2 B(t)$$

and the estimates (5.13), (5.15) remain valid. The proof is identical. This remark will be used in Sections 6 and 7.

Proof. We only have to show the estimates (5.13) and (5.15). These are straightforward calculations. We use the following notation,

$$a\sigma + i\theta = \gamma e^{i\phi} \tag{5.17}$$

(i.e., $\theta = \gamma \sin \phi$, $a\sigma = \gamma \cos \phi$ and $\gamma^2 = \theta^2 + a^2\sigma^2$). Hence, from (5.8),

$$\eta_{\theta,\sigma}(t) = \frac{\gamma}{2a}\left(e^{i(\phi - at)} + e^{-i(\phi - at)}\right) \tag{5.18}$$

and

$$H(t,\omega) = \frac{\gamma^2}{4a^2}\left| e^{2i\phi} \frac{e^{-it(a+\omega)} - 1}{a+\omega} - \frac{e^{it(a-\omega)} - 1}{a-\omega}\right|^2 \tag{5.19}$$

which proves (5.13).

To prove (5.14)–(5.15), for any $\Omega > 1$ we obtain, by extracting the worst singularity

$$\begin{aligned} Q(t) &= \lambda^2 \int_0^\Omega \frac{\omega(\cosh \beta\omega + 1)}{2 \sinh \beta\omega} H(t,\omega)\, d\omega \\ &= \lambda^2 \frac{\gamma^2}{4a^2} \widetilde{B}(t) + \lambda^2 \frac{\gamma^2}{4a^2} \int_0^\Omega \frac{\omega(\cosh \beta\omega + 1)}{2 \sinh \beta\omega} \left| \frac{e^{it(a-\omega)} - 1}{a-\omega}\right|^2 d\omega \end{aligned} \tag{5.20}$$

with

$$\begin{aligned} \widetilde{B}(t) := \int_0^\Omega \frac{\omega(\cosh \beta\omega + 1)}{2 \sinh \beta\omega} \Bigg\{ &\left| \frac{e^{-it(a+\omega)} - 1}{a+\omega}\right|^2 \\ &- 2\,\mathrm{Re}\left(e^{2i\phi} \frac{e^{-it(a+\omega)} - 1}{a+\omega} \frac{e^{it(a-\omega)} - 1}{a-\omega}\right)\Bigg\}\, d\omega \end{aligned} \tag{5.21}$$

and $\widetilde{B}(0)=0$. With the substitution $\omega'=t(a-\omega)$ in (5.21), one easily computes

$$|\widetilde{B}(t)| \leqslant C[1+\beta^{-1}][1+(\log t)_+][1+\log \Omega] \tag{5.22}$$

The second integral in (5.20) is proportional to t for large t since $\Omega>1$. Obviously it becomes uniformly bounded if $\Omega<a\leqslant 1$ (a trivial behaviour), and this is the very reason why we assumed $\Omega>1$ in this section. Then the main contribution comes from $\omega \sim a$, and by the same change of variables as above, the result is,

$$Q(t)=\lambda^2\gamma^2 B(t)+I\lambda^2 t\gamma^2 \frac{\cosh a\beta+1}{2a \sinh a\beta} \tag{5.23}$$

with $I:=\pi/2$, and $\widetilde{B}(t)$ is replaced by some $B(t)$ which also satisfies (5.22) and $B(0)=0$. ∎

5.3. The Fokker–Planck Equation in the Caldeira–Leggett Limits

In this section we rigorously perform the scaling limit introduced in ref. 5. We prove the following,

Theorem 5.1. Let w_A^ε be the Wigner distribution of the test-particle after the thermodynamic limit, as given by Lemma 4.3. We recall that ε stands for (β, Ω, λ). Let $\lambda=\lambda_0\beta^{1/2}$, λ_0 fixed.

(a) [Nonzero frequency shift] Assume that $a^2=1-\lambda^2\Omega=1-\lambda_0^2\beta\Omega \in (0, 1]$ is fixed. Then for any $t\geqslant 0$ the following weak limit exists

$$W(t, x, v)=\lim_{\substack{\Omega\to\infty,\, \beta\to 0\\ \beta\Omega=(1-a^2)\,\lambda_0^{-2}}} w_A^\varepsilon(t, x, v) \tag{5.24}$$

The limit holds in the topology of $C^0([0, \infty)_t; \mathscr{D}'_{x,v})$. Moreover, W satisfies the Fokker–Planck equation,

$$\partial_t W+v\partial_x W-a^2x\partial_v W-\frac{\lambda_0^2\pi}{2}\Delta_v W=0 \tag{5.25}$$

with initial datum $W(t=0)=w_0$ satisfying (4.10).

(b) [No frequency shift] For any $t \geqslant 0$ the following weak limit exists,

$$W(t, x, v) = \lim_{\Omega \to \infty} \lim_{\beta \to 0} w_A^\varepsilon(t, x, v) \tag{5.26}$$

[the order of limits cannot be interchanged], and W satisfies the Fokker–Planck equation,

$$\partial_t W + v\partial_x W - x\partial_v W - \frac{\lambda_0^2 \pi}{2} \Delta_v W = 0 \tag{5.27}$$

with initial datum $W(t=0) = w_0$ satisfying (4.10).

Proof. For the proof of part (a) first notice that Lemma 5.1 applies since the right-hand side of (5.3) goes to zero under the prescribed limits. Hence X can be replaced by $\tilde{X}$ and we can therefore rely on Lemma 5.2 above. On the other hand, since we assumed $\lambda = \lambda_0 \beta^{1/2}$, we readily observe,

$$\lim{}^* Q(t) = \lambda_0^2 \lim{}^* \int_0^\Omega \beta A_\beta^2(\omega)\, H(t, \omega)\, d\omega = \lambda_0^2 \int_0^\infty \left| \int_0^t \eta_{\theta,\sigma}^2(s)\, e^{-i\omega s}\, ds \right|^2 d\omega \tag{5.28}$$

where $\lim^*$ stands for the simultaneous limit $\beta \to 0$, $\Omega \to \infty$ such that $a^2 = 1 - \lambda_0^2 \beta \Omega \in (0, 1]$ is fixed. Here we used that $\beta A_\beta(\omega)^2 \to 1$ in our limit if $\omega \leqslant \Omega^{1/2}$ and that $H(t, \omega) \in L^1(d\omega)$, see (5.13). The contribution $\omega \geqslant \Omega^{1/2}$ to the integral vanishes in the limit by the estimate (5.13) and the trivial bound $(z \cosh z + 1)/\sinh z \leqslant 2(1+z)$. Hence from the unitarity of the Fourier transform

$$\int_0^\infty \left| \int_0^t g(s)\, e^{-i\omega s}\, ds \right|^2 d\omega = \pi \int_0^t |g(s)|^2\, ds \tag{5.29}$$

which is valid for any real function g, we obtain

$$\lim{}^* Q(t) = \lambda_0^2 \pi \int_0^t \eta_{\theta,\sigma}^2(s)\, ds \tag{5.30}$$

Here $t \geqslant 0$ is used, and this step is the origin of irreversibility. The end of the calculation is trivial. From Lemma 5.2 together with (5.30) we have,

$$\begin{aligned} &\lim{}^* \int_{\mathbb{R}^2} w_A^\varepsilon(t, x, v)\, \overline{\phi(x, v)}\, dx\, dv \\ &\quad = \int_{\mathbb{R}^2} \hat{w}_0(\xi_{\theta,\sigma}(t), \eta_{\theta,\sigma}(t))\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{-I\lambda_0^2 \int_0^t \eta_{\theta,\sigma}^2(s)\, ds}\, d\theta\, d\sigma \end{aligned} \tag{5.31}$$

where η and ξ are defined in (5.8) and $I=\pi/2$. We can define,

$$W(t, x, v) := \lim{}^* w_A^\varepsilon(t, x, v) \tag{5.32}$$

as a weak limit given by (5.31). Then differentiating (5.31) gives (using (5.8)),

$$\begin{aligned}
&\int_{\mathbb{R}^2} \partial_t W(t, x, v)\, \overline{\phi(x, v)}\, dx\, dv \\
&\quad = \int_{\mathbb{R}^2} \partial_t \hat{W}(t, \theta, \sigma)\, \overline{\hat{\phi}(\theta, \sigma)}\, d\theta\, d\sigma \\
&\quad = \int_{\mathbb{R}^2} [-a^2 \eta_{\theta,\sigma}(t)\, \partial_\xi + \xi_{\theta,\sigma}(t)\, \partial_\eta - I\lambda_0^2 \eta_{\theta,\sigma}^2(t)] \\
&\qquad \times \hat{w}_0(\xi_{\theta,\sigma}(t), \eta_{\theta,\sigma}(t))\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{-I\lambda_0^2 \int_0^t \eta_{\theta,\sigma}^2(s)\, ds}\, d\theta\, d\sigma
\end{aligned} \tag{5.33}$$

Letting $t=0$, we have,

$$\partial_t|_{t=0} \hat{W}(t, \theta, \sigma) = [-a^2 \sigma \partial_\theta + \theta \partial_\sigma - I\lambda_0^2 \sigma^2]\, \hat{W}(t, \theta, \sigma)|_{t=0} \tag{5.34}$$

which is exactly the Fokker–Planck equation (5.27) after Fourier transforming,

$$\partial_t|_{t=0} W(t, x, v) = [a^2 x \partial_v - v \partial_x + I\lambda_0^2 \Delta_v]\, W(t, x, v)|_{t=0} \tag{5.35}$$

Considering $t=0$ is not a restriction, since the proof works for any L^1 initial condition.

The proof of part (b) is completely analogous. We again notice that under the prescribed limits the right-hand side of (5.3) goes to zero, hence Lemma 5.1 applies. Here $\eta_{\theta,\sigma}$ and $\xi_{\theta,\sigma}$ depend on the limiting parameters, since $a^2 = 1 - \lambda^2 \Omega = 1 - \lambda_0^2 \beta \Omega$. But $\lim_{\beta \to 0} a = 1$, hence

$$\lim_{\beta \to 0} \eta_{\theta,\sigma}(s) = \theta \sin s + \sigma \cos s, \qquad \lim_{\beta \to 0} \xi_{\theta,\sigma}(s) = \theta \cos s - \sigma \sin s \tag{5.36}$$

uniformly for $s \in [0, t]$. Therefore

$$\begin{aligned}
\lim_{\Omega \to \infty} \lim_{\beta \to 0} Q(t) &= \lambda_0^2 \lim_{\Omega \to \infty} \int_0^\Omega \left| \int_0^t [\theta \sin s + \sigma \cos s]\, e^{-i\omega s}\, ds \right|^2 d\omega \\
&= \lambda_0^2 \int_0^\infty \left| \int_0^t [\theta \sin s + \sigma \cos s]\, e^{-i\omega s}\, ds \right|^2 d\omega \\
&= \pi \lambda_0^2 \int_0^t [\theta \sin s + \sigma \cos s]^2\, ds
\end{aligned} \tag{5.37}$$

Again, the last step is robust in a sense that it does not use the particular form of the function $[\theta \sin s + \sigma \cos s]$, instead it uses (5.29). But it is rigid in a sense that $\Omega = \infty$ is essential to get diffusive (linear) behaviour for the mean square displacement (5.16).

To conclude, we follow the calculation (5.31)–(5.35). In addition to the limit (5.37), we have to replace $\xi_{\theta,\sigma}(s), \eta_{\theta,\sigma}(s)$ by their limiting values (5.36) in the argument of $\hat{w}_0$ to arrive at the analogue of (5.31). Dominated convergence theorem applies if we assume, additionally, that $\hat{w}_0$ is continuous and bounded. However $\hat{w}_0 \in L^1$, hence it can be approximated by such functions in L^1-norm. Using that the flow $(\theta, \sigma) \mapsto (\xi_{\theta,\sigma}(s), \eta_{\theta,\sigma}(s))$ is measure preserving and that $\hat{\phi}$ is bounded, one can easily see that the approximation error can be made arbitrarily small.

The rest of the calculation is identical to the proof of part (a) and we obtain (5.27). ∎

6. SCALING LIMIT AT HIGH TEMPERATURE: THE FRICTIONLESS HEAT EQUATION

We propose a different way to get diffusion from the Hamiltonian (3.1). As we mentioned, obtaining diffusion for the test-particle means that we have to extract linear dependence in time for $Q(t)$. In this section, linear growth is obtained from time rescaling and from the special form of linear combinations of $\sin s$ and $\cos s$ in Lemma 5.2. It relies on a resonance effect which comes from a singularity near $\omega \sim a$. The system $\tilde{X}''(t) + a^2 \tilde{X}(t)$ (see (5.1)) picks up those modes from the forcing term $f(t)$ in (4.34) for which the frequency ω is close to its eigenfrequency. So, in this section we assume $\Omega > 1$ but finite, contrary to the previous section.

This effect is more robust (see the remark after (5.37)) in the sense that one *could* leave the hyperbolic functions βA_β^2 in (5.28) without ensuring a limit where it goes to 1. In other terms, we do not need the high temperature limit $\beta \to 0$ to obtain diffusion, unlike in Section 5.3, where this limit made the $d\omega$ measure uniform and we recovered a white noise forcing term.

Nevertheless, Lemma 5.2 needs the right-hand side of (5.3) to go to zero in order to be applicable (one needs the friction to vanish), and this cannot be achieved keeping β fixed (Section 3.), hence we again set $\lambda = \lambda_0 \beta^{1/2}$, $\beta \to 0$.

6.1. Large Space/Time Convergence of the Wigner Distribution

Let α be a small parameter. We describe the behaviour of the test-particle, as given by its Wigner distribution w_A^ε on time scales of order $1/\alpha^2$.

We consider the diffusive scaling, i.e., the space coordinate scales as $1/\alpha$. Since the test-particle is a fast harmonic oscillator, and energies are transferred back and forth between space and velocity, we also have to consider velocities of order $1/\alpha$. Hence we introduce the following scaling,

$$t = T\alpha^{-2}, \qquad x = X\alpha^{-1}, \qquad v = V\alpha^{-1} \tag{6.1}$$

where the capital letters are unscaled quantities (macroscopic variables). The rescaled reduced Wigner transform is defined as,

$$W_T^{\varepsilon,\alpha}(X, V) := w_A^{\varepsilon}(T\alpha^{-2}, X\alpha^{-1}, V\alpha^{-1}) \tag{6.2}$$

where w_A^{ε} is defined in Lemma 4.3 (after the thermodynamic limit). Its Fourier transform is,

$$\hat{W}_T^{\varepsilon,\alpha}(\Theta, \Sigma) = \alpha^2 \hat{w}_A^{\varepsilon}(T\alpha^{-2}, \Theta\alpha, \Sigma\alpha) \tag{6.3}$$

where we use $\Theta = \theta\alpha^{-1}$ and $\Sigma = \sigma\alpha^{-1}$ rescaled dual variables. The initial condition is,

$$W_{T=0}^{\varepsilon,\alpha}(X, V) = W_0(X, V), \qquad \hat{W}_{T=0}^{\varepsilon,\alpha}(\Theta, \Sigma) = \hat{W}_0(\Theta, \Sigma) \tag{6.4}$$

and we assume that

$$\hat{W}_0(\Theta, \Sigma) \in L^1(\mathbb{R}_\Theta \times \mathbb{R}_\Sigma) \tag{6.5}$$

The macroscopic testfunction $\Phi(X, V)$ is a smooth function with compact support, the microscopic testfunction is defined as,

$$\phi(x, v) = \Phi(x\alpha, v\alpha) = \Phi(X, V) \tag{6.6}$$

and in Fourier variables, $\hat{\phi}(\theta, \sigma) = \alpha^{-2}\hat{\Phi}(\theta\alpha^{-1}, \sigma\alpha^{-1}) = \alpha^{-2}\hat{\Phi}(\Theta, \Sigma)$.

We are now in position to state the theorem of this section,

Theorem 6.1. Define the large time/space scale Wigner distribution $W_T^{\varepsilon,\alpha}(X, V)$ as in (6.2). Assume (6.5) for the initial data, let $\lambda = \lambda_0\beta^{1/2}$ with a fixed $\lambda_0 > 0$ and fix the frequency cutoff $\Omega > 1$. Hence the limits of the parameters $\varepsilon = (\beta, \Omega, \lambda)$ are reduced to $\beta \to 0$. Then:

(a) The following high-temperature limit exists in the weak sense for any $T \geqslant 0$:

$$W_T^{\alpha}(X, V) := \lim_{\beta \to 0} W_T^{\varepsilon,\alpha}(X, V) \tag{6.7}$$

(b) Define the following time average of W^α over one cycle of the harmonic oscillator (5.8),

$$W_T^{\#,\alpha}(X, V) := \frac{1}{2\pi\alpha^2} \int_T^{T+2\pi\alpha^2} W_S^\alpha(X, V)\, dS \tag{6.8}$$

Then the weak limit,

$$W_T^+(X, V) := \lim_{\alpha \to 0} W_T^{\#,\alpha}(X, V) \tag{6.9}$$

exists for each $T \geqslant 0$ and it satisfies the heat equation in phase space,

$$\partial_T W_T^+ = \frac{\pi\lambda_0^2}{4} (\Delta_X + \Delta_V)\, W_T^+ \tag{6.10}$$

with initial condition $W_{T=0}^+(X, V)$ given by

$$\hat{W}_0^+(X, V) = \frac{1}{2\pi} \int_0^{2\pi} \hat{W}_0(X \sin s + V \cos s, X \cos s - V \sin s)\, ds \tag{6.11}$$

(c) Define the radial average,

$$W_T^{*,\alpha}(X, V) := \frac{1}{2\pi} \int_0^{2\pi} W_T^\alpha(R \cos s, R \sin s)\, ds \tag{6.12}$$

with $R := \sqrt{X^2 + V^2}$, and clearly $W_T^{*,\alpha}$ depends on R only. Again, the weak limit,

$$W_T^\dagger(X, V) := \lim_{\alpha \to 0} W_T^{*,\alpha}(X, V) \tag{6.13}$$

exists and the radially symmetric function $W_T^\dagger$ satisfies the heat equation (6.10) with initial condition,

$$W_{T=0}^\dagger(X, V) := \frac{1}{2\pi} \int_0^{2\pi} W_0(R \cos s, R \sin s)\, ds$$

Remark 1. The same theorem is true if the frequency distribution function $\varrho(\omega)$ is not uniform (see (1.4)), but it is only bounded and with bounded derivative. In particular the sharp cutoff is not necessary. The right-hand side of the equation (6.10) is multiplied by the resonant spectral density $\varrho(1)$. The proof relies on two modifications of the $\varrho \equiv 1$ proof given below. First, the memory kernel $M(t)$ (see (4.20) and (4.29)) is modified to $\lambda^2 \int_0^{\Omega} \cos(\omega t)\, \varrho(\omega)\, d\omega$, and it still satisfies an estimate similar to (4.28) which leads to Lemma 5.1, hence the memory can be eliminated. Second, Remark 2 after Lemma 5.2 gives the large time behavior of $Q(t)$ in the general case. The details are left to the reader.

Remark 2. Here we identified the equation in a weak sense in the space and velocity variables, but in a strong sense in the time variable and some averaging ((6.8) or (6.12)) was needed to ensure the existence of the limit. If we want to consider the limit in a weak sense in time as well, then there is no need for averaging. Based upon part (b), one can easily prove that $W_T^+(X, V)$ can also be identified as the weak limit in space, velocity and time, i.e., we have

Corollary 6.1. Under the above conditions the weak limit

$$W_T^+(X, V) := \lim_{\alpha \to 0} \lim_{\beta \to 0} W_T^{\varepsilon, \alpha}(X, V)$$

exists in the topology of $\mathscr{D}'([0, \infty)_T \times \mathbb{R}_X \times \mathbb{R}_V)$, it coincides with (6.9) and satisfies (6.10).

Proof of Theorem 6.1. Using the rescaling and the definition of w_A^ε (4.41), we have,

$$\begin{aligned}
\langle W_T^{\varepsilon,\alpha}, \Phi \rangle &= \int_{\mathbb{R}^2} W_T^{\varepsilon,\alpha}(X, V)\, \overline{\Phi(X, V)}\, dX\, dV \\
&= \alpha^2 \int_{\mathbb{R}^2} w_A^\varepsilon(T\alpha^{-2}, x, v)\, \overline{\phi(x, v)}\, dx\, dv \\
&= \alpha^2 \mathbf{E} \int_{\mathbb{R}^6} \hat{w}_0(\xi, \eta)\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{i(x\xi + v\eta)} \\
&\quad \times e^{-i(\theta X(t) + \sigma X'(t))}\, d\xi\, d\eta\, dx\, dv\, d\theta\, d\sigma \\
&= \mathbf{E} \int_{\mathbb{R}^6} \hat{W}_0(\xi\alpha^{-1}, \eta\alpha^{-1})\, \overline{\hat{\Phi}(\Theta, \Sigma)} \\
&\quad \times\, e^{i(x\xi + v\eta)} e^{-i\alpha(\Theta X(t) + \Sigma X'(t))}\, d\xi\, d\eta\, dx\, dv\, d\Theta\, d\Sigma
\end{aligned} \tag{6.14}$$

where $t = T\alpha^{-2}$.

First Step: *the limit* $\beta \to 0$.

Due to the choice $\lambda = \lambda_0 \beta^{1/2}$, we can replace $X(t)$ by $\tilde{X}(t)$ in the $\beta \to 0$ limit. For, the right-hand side of (5.3) goes to zero as $\beta \to 0$, hence Lemma 5.1 applies. Hence,

$$\lim_{\beta \to 0} \langle W_T^{\varepsilon,\alpha}, \Phi \rangle = \lim_{\beta \to 0} \mathbf{E} \int_{\mathbb{R}^6} \hat{W}_0(\xi\alpha^{-1}, \eta\alpha^{-1}) \overline{\hat{\Phi}(\Theta, \Sigma)} \times e^{i(x\xi + v\eta)} e^{-i\alpha(\Theta \tilde{X}(t) + \Sigma \tilde{X}'(t))} \, d\xi \, d\eta \, dx \, dv \, d\Theta \, d\Sigma$$

$$= \lim_{\beta \to 0} \mathbf{E} \int_{\mathbb{R}^2} \hat{W}_0(\xi_{\Theta,\Sigma}(T\alpha^{-2}), \eta_{\Theta,\Sigma}(T\alpha^{-2})) \times \overline{\hat{\Phi}(\Theta, \Sigma)} \, e^{-1/2 Q(T\alpha^{-2})} \, d\Theta \, d\Sigma \tag{6.15}$$

where in the second step we also used Lemma 5.2 and the fact that $\alpha^{-1}\xi_{\alpha\Theta,\alpha\Sigma} = \xi_{\Theta,\Sigma}$ and $\alpha^{-1}\eta_{\alpha\Theta,\alpha\Sigma} = \eta_{\Theta,\Sigma}$ (see (5.8)).

Recall that both $Q(t)$ and the trajectories $\xi_{\Theta,\Sigma}, \eta_{\Theta,\Sigma}$ depend on β, since $a^2 = 1 - \lambda^2 \Omega = 1 - \lambda_0^2 \beta \Omega$ appears in their definition (see (5.8)). Similarly to the argument at the end of the proof of part (b) of Theorem 5.1, using that $\hat{W}_0 \in L^1(d\Theta \, d\Sigma)$, $\hat{\Phi} \in L^\infty \cap C^0$, $Q \geqslant 0$, we see that the limit can be taken inside the integral and the trajectories $\xi_{\Theta,\Sigma}, \eta_{\Theta,\Sigma}$ can be replaced by their limiting values (as $a \to 1$)

$$\eta^*_{\Theta,\Sigma}(s) := \theta \sin t + \sigma \cos t \qquad \xi^*_{\Theta,\Sigma}(s) := \theta \cos t - \sigma \sin t \tag{6.16}$$

We also use (see (5.14)) that

$$\lim_{\beta \to 0} Q(t) = I\lambda_0^2 t\gamma^2 + \lambda_0^2 \gamma^2 B_0(t) \tag{6.17}$$

with $B_0(t)$ satisfying $B_0(0) = 0$ and

$$|B_0(t)| \leqslant C[1 + (\log t)_+][1 + \log \Omega] \tag{6.18}$$

(see (5.15)). We also recall that $\gamma^2 = \theta^2 + \sigma^2 = \alpha^2(\Theta^2 + \Sigma^2) =: \alpha^2 \Gamma^2$. Hence,

$$\lim_{\beta \to 0} \langle W_T^{\varepsilon,\alpha}, \Phi \rangle = \int_{\mathbb{R}^2} \hat{W}_0(\xi^*_{\Theta,\Sigma}(T\alpha^{-2}), \eta^*_{\Theta,\Sigma}(T\alpha^{-2})) \overline{\hat{\Phi}(\Theta, \Sigma)} \times \exp\{-\tfrac{1}{2}[I\lambda_0^2 T\alpha^{-2} + \lambda_0^2 B_0(T\alpha^{-2})]\alpha^2(\Theta^2 + \Sigma^2)\} \, d\Theta \, d\Sigma \tag{6.19}$$

This relation defines the Fourier transform,

$$\hat{W}_T^{\alpha}(\Theta, \Sigma) := \lim_{\beta \to 0} \hat{W}_T^{\varepsilon,\alpha}(\Theta, \Sigma) \tag{6.20}$$

as a weak limit, and its inverse Fourier transform,

$$W_T^{\alpha}(X, V) := \lim_{\beta \to 0} W_T^{\varepsilon, \alpha}(X, V)$$

We can compute its time derivative in Fourier space,

$$\begin{aligned}\langle \partial_T \hat{W}_T^{\alpha}, \hat{\Phi} \rangle = \int \alpha^{-2} \Big[& -\eta^*_{\Theta, \Sigma}(T\alpha^{-2})\, \partial_\xi + \xi^*_{\Theta, \Sigma}(T\alpha^{-2})\, \partial_\eta \\ & - \frac{\alpha^2}{2} [I\lambda_0^2 + \lambda_0^2 B_0'(T\alpha^{-2})]\, (\Theta^2 + \Sigma^2) \Big] \\ & \times \hat{W}_0(\xi^*_{\Theta, \Sigma}(T\alpha^{-2}), \eta^*_{\Theta, \Sigma}(T\alpha^{-2}))\, \overline{\hat{\Phi}(\Theta, \Sigma)} \\ & \times \exp\left\{ -\frac{1}{2} [I\lambda_0^2 T\alpha^{-2} + \lambda_0^2 B_0(T\alpha^{-2})]\, \alpha^2(\Theta^2 + \Sigma^2) \right\} d\Theta\, d\Sigma \end{aligned} \tag{6.21}$$

As usual, we can let $T = 0$ to obtain,

$$\begin{aligned}&\partial_T|_{T=0}\, \hat{W}_T^{\alpha}(\Theta, \Sigma) \\ &\quad = \alpha^{-2} \left[-\Sigma\partial_\Theta + \Theta\partial_\Sigma - \frac{\alpha^2}{2} [I\lambda_0^2 + \lambda_0^2 B_0'(0)](\Theta^2 + \Sigma^2) \right] \hat{W}_0(\Theta, \Sigma)\end{aligned} \tag{6.22}$$

Second Step: *the macroscopic limit* $\alpha \to 0$.

Now the difficulty in (6.22) is that the convective term is too big compared to the last diffusive term since the motion takes place on two different time scales. There is the fast (microscopic) time scale of the harmonic oscillator described by $\alpha^{-2}[-\Sigma\partial_\Theta + \Theta\partial_\Sigma]$. Then there is a slow, macroscopic diffusive scale. We present two ways to average out the fast motion.

Part (b) of Theorem 6.1: Averaging over a cycle.

Here we define $W^{\#, \alpha}$ according to (6.8). Now for any T fixed the formula,

$$\begin{aligned}\lim_{\alpha \to 0} \langle \hat{W}_T^{\#, \alpha}, \hat{\Phi} \rangle &= \lim_{\alpha \to 0} \int \hat{W}_T^{\#, \alpha}(\Theta, \Sigma)\, \overline{\hat{\Phi}(\Theta, \Sigma)}\, d\Theta\, d\Sigma \\ &= \lim_{\alpha \to 0} \int \Bigg[\frac{1}{2\pi\alpha^2} \int_T^{T + 2\pi\alpha^2} \hat{W}_0(\xi^*_{\Theta, \Sigma}(S\alpha^{-2}), \eta^*_{\Theta, \Sigma}(S\alpha^{-2})) \\ &\qquad \times e^{-I_1\lambda_0^2 S(\Theta^2 + \Sigma^2)}\, dS \Bigg]\, \overline{\hat{\Phi}(\Theta, \Sigma)}\, d\Theta\, d\Sigma \end{aligned} \tag{6.23}$$

defines a function,

$$\hat{W}_T^{+}(\Theta, \Sigma) := \lim_{\alpha \to 0} \hat{W}_T^{\#,\alpha}(\Theta, \Sigma) \tag{6.24}$$

weakly, as we show below. Here $I_1 := I/2 = \pi/4$ for brevity. Note that in (6.23) we neglected the term involving B_0 in the exponential (see (6.19)) since the estimate (6.18) readily implies $\alpha^2 B_0(T\alpha^{-2}) \to 0$. The exponential factor in (6.19) converges to that in (6.23) uniformly for all $S \leqslant T$. Using $\hat{\Phi} \in L^1$, we can apply the dominated convergence theorem along with approximating $\hat{W}_0$ by bounded functions, similarly to the argument at the end of the proof of Theorem 5.1.

We have to show that the limit on the right-hand-side of (6.23) exists,

$$\begin{aligned}
&\langle \hat{W}_T^{\#,\alpha}, \hat{\Phi} \rangle \\
&\quad = \int_{\mathbb{R}^2} \bigg[\frac{1}{2\pi\alpha^2} \int_T^{T+2\pi\alpha^2} \hat{W}_0(\xi^*_{\Theta,\Sigma}(S\alpha^{-2}), \eta^*_{\Theta,\Sigma}(S\alpha^{-2}))\, e^{-I_1\lambda_0^2 T(\Theta^2+\Sigma^2)}\, dS \\
&\qquad + \frac{1}{2\pi\alpha^2} \int_T^{T+2\pi\alpha^2} \hat{W}_0(\xi^*_{\Theta,\Sigma}(S\alpha^{-2}), \eta^*_{\Theta,\Sigma}(S\alpha^{-2})) \\
&\qquad \times [e^{-I_1\lambda_0^2 S(\Theta^2+\Sigma^2)} - e^{-I_1\lambda_0^2 T(\Theta^2+\Sigma^2)}]\, dS \bigg]\, \overline{\hat{\Phi}(\Theta, \Sigma)}\, d\Theta\, d\Sigma
\end{aligned} \tag{6.25}$$

The first term in (6.25) is independent of α, as it is just the integral of $\hat{W}_0(\xi^*(s), \eta^*(s))$ over one full cycle of the harmonic oscillator (6.16),

$$\begin{aligned}
&\frac{1}{2\pi\alpha^2} \int_T^{T+2\pi\alpha^2} \hat{W}_0(\xi^*_{\Theta,\Sigma}(S\alpha^{-2}), \eta^*_{\Theta,\Sigma}(S\alpha^{-2}))\, dS \\
&\qquad = \frac{1}{2\pi} \int_0^{2\pi} \hat{W}_0(\xi^*_{\Theta,\Sigma}(s), \eta^*_{\Theta,\Sigma}(s))\, ds
\end{aligned} \tag{6.26}$$

The second term in (6.25) vanishes in the limit $\alpha \to 0$ since,

$$\begin{aligned}
&|e^{-I_1\lambda_0^2 S(\Theta^2+\Sigma^2)} - e^{-I_1\lambda_0^2 T(\Theta^2+\Sigma^2)}| \\
&\qquad \leqslant 2\pi I_1 \lambda_0 \alpha^2(\Theta^2 + \Sigma^2)\, e^{-I_1\lambda_0^2 T(\Theta^2+\Sigma^2)}
\end{aligned} \tag{6.27}$$

(use that $|S - T| \leqslant 2\pi\alpha^2$), which kills the factor α^{-2} in (6.25) and then the length of the integration interval goes to zero. Dominated convergence theorem again has to be applied after an approximation. This shows that the limit in (6.24) makes sense and,

$$\begin{aligned}\langle W_T^+, \Phi \rangle &= \langle \hat{W}_T^+, \hat{\Phi} \rangle \\ &= \int_{\mathbb{R}^2} \left[\frac{1}{2\pi} \int_0^{2\pi} \hat{W}_0(\xi^*_{\Theta,\Sigma}(s), \eta^*_{\Theta,\Sigma}(s))\, ds \right] \\ &\quad \times e^{-I_1 \lambda_0^2 T(\Theta^2+\Sigma^2)}\, \overline{\hat{\Phi}(\Theta,\Sigma)}\, d\Theta\, d\Sigma \end{aligned} \tag{6.28}$$

The time derivative is,

$$\begin{aligned}\langle \partial_T W_T^+, \Phi \rangle &= -I_1 \lambda_0^2 \int_{\mathbb{R}^2} (\Theta^2+\Sigma^2) \left[\frac{1}{2\pi} \int_0^{2\pi} \hat{W}_0(\xi^*_{\Theta,\Sigma}(s), \eta^*_{\Theta,\Sigma}(s))\, ds \right] \\ &\quad \times e^{-I_1 \lambda_0^2 T(\Theta^2+\Sigma^2)}\, \overline{\hat{\Phi}(\Theta,\Sigma)}\, d\Theta\, d\Sigma \\ &= -I_1 \lambda_0^2 \langle \hat{W}_T^+, (\Theta^2+\Sigma^2)\, \hat{\Phi} \rangle \\ &= -I_1 \lambda_0^2 \langle W_T^+, -(\Delta_X + \Delta_V)\, \Phi \rangle \end{aligned} \tag{6.29}$$

which completes the proof of (6.10). The initial condition (6.11) is easily obtained from (6.28) by setting $T=0$ and taking inverse Fourier transform.

Part (c) of Theorem 6.1: Radial average

The other possibility to eliminate the fast modes is to use the radial function $W_T^{*,\alpha}$ defined in (6.12). Now the formula,

$$\begin{aligned}&\lim_{\alpha \to 0} \langle \hat{W}_T^{*,\alpha}, \hat{\Phi} \rangle \\ &= \lim_{\alpha \to 0} \int \hat{W}_T^{*,\alpha}(\Theta,\Sigma)\, \overline{\hat{\Phi}(\Theta,\Sigma)}\, d\Theta\, d\Sigma \\ &= \lim_{\alpha \to 0} \int \left[\frac{1}{2\pi} \int_0^{2\pi} \hat{W}_0(\xi^*_{\Gamma\cos s, \Gamma \sin s}(T\alpha^{-2}), \eta^*_{\Gamma\cos s, \Gamma\sin s}(T\alpha^{-2}))\, ds \right] \\ &\quad \times e^{-I_1 \lambda_0^2 T(\Theta^2+\Sigma^2)}\, \overline{\hat{\Phi}(\Theta,\Sigma)}\, d\Theta\, d\Sigma \end{aligned} \tag{6.30}$$

(with $\Gamma := \sqrt{\Theta^2+\Sigma^2}$) defines a radial function,

$$\hat{W}_T^\dagger(\Theta,\Sigma) := \lim_{\alpha \to 0} \hat{W}_T^{*,\alpha}(\Theta,\Sigma) \tag{6.31}$$

(depending only on $\Theta^2+\Sigma^2$) as a weak limit, as we show below. Note that in (6.30) we again neglected the term involving B_0 in the exponential for the same reason as in (6.23).

We have to show that the limit on the right-hand-side of (6.30) exists. But,

$$\xi^*_{\Gamma\cos s,\,\Gamma\sin s}(T\alpha^{-2})=\Gamma\cos(s+T\alpha^{-2}),$$
$$\eta^*_{\Gamma\cos s,\,\Gamma\sin s}(T\alpha^{-2})=\Gamma\sin(s+T\alpha^{-2})$$

hence,

$$\frac{1}{2\pi}\int_0^{2\pi}\hat{W}_0(\xi^*_{\Gamma\cos s,\,\Gamma\sin s}(T\alpha^{-2}),\eta^*_{\Gamma\cos s,\,\Gamma\sin s}(T\alpha^{-2}))\,ds$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\hat{W}_0(\Gamma\cos s,\Gamma\sin s)\,ds=:\hat{W}_0^{\dagger}(\Theta,\Sigma)$$

independently of α, which is the "radialized" initial condition in Fourier space.

So it is clear that the limit on the right-hand-side of (6.30) exists,

$$\lim_{\alpha\to 0}\langle\hat{W}_T^{*,\alpha},\hat{\Phi}\rangle=\int\hat{W}_0^{\dagger}(\Theta,\Sigma)\,e^{-I_1\lambda_0^2T(\Theta^2+\Sigma^2)}\,\overline{\hat{\Phi}(\Theta,\Sigma)}\,d\Theta\,d\Sigma=:\langle\hat{W}_T^{\dagger},\hat{\Phi}\rangle$$

and clearly $W_T^{\dagger}$ also satisfies the heat equation (6.1). This ends the proof of Theorem 6.1. ∎

7. HEAT EQUATION WITH FRICTION AT FINITE TEMPERATURE

Here we choose a scaling where the Markovian part of the friction term does not vanish, i.e., we can keep β fixed and still get finite diffusion. Again we look at large time $t=T\delta^{-1}$ but now we do not scale the space variable. To eliminate the fast mode, we again integrate the angle. The result is a radial Fokker–Planck equation with friction. While the test-particle performs many cycles, it slowly diffuses out, and this diffusion is slowed down by a friction. The diffusion comes from resonance.

In this scaling limit the solution of (4.36) is close to the solution $\tilde{X}(t)$ of an equation without a time delayed (non-Markovian) friction term, but a Markovian friction term will be present. Let us choose,

$$\lambda:=\lambda_0\delta^{1/2} \tag{7.1}$$

with some $\lambda_0 < 1$ fixed. We compare the solution of (4.36) to that of

$$\widetilde{X}''(t) + I\lambda^2\widetilde{X}'(t) + a^2\widetilde{X}(t) = f(t); \qquad \widetilde{X}(0) = x, \quad \widetilde{X}'(0) = v \tag{7.2}$$

with $a^2 := 1 - \lambda^2\Omega = 1 - \lambda_0^2\delta^{-1}\Omega$, and,

$$I = \int_0^\infty \frac{\sin \Omega s}{s}\, ds = \frac{\pi}{2} \tag{7.3}$$

We choose the scaling such that $a \in (0, 1]$, hence we always assume that $\Omega \leqslant \delta^{-1}$, but to exploit resonance, we also assume $\Omega > 2$. The new term $\lambda^2 I\widetilde{X}'(t)$ for the approximate characteristic is due to the fact that $M(t) \sim \lambda^2 I\delta_0(t)$ as $\Omega \to 0$, where δ_0 denotes the Dirac delta measure. This term is the main part of the full friction $(M \star X')$ in (4.36). Notice that it is small compared with the pure harmonic oscillator terms, $\widetilde{X}'' + a^2\widetilde{X}$, but it is not negligible, since we will consider long times $t \sim \lambda^{-2}$.

7.1. A Priori Bounds and Continuity Results

As in Section 5.1 we need a priori estimates for X, i.e., for,

$$F(t) := \sup_{s \leqslant t} \mathbf{E}\,|X(s)| + \sup_{s \leqslant t} \mathbf{E}\,|X'(s)|$$

and estimates on the difference between $\widetilde{X}(t)$ and $X(t)$. The estimate (4.37) in Lemma 4.2 (which originates in (4.24) in Lemma 4.1), however, is not precise enough for large times. The following estimate is a more precise version of Lemma 4.2.

Lemma 7.1. Let $t = T\delta^{-1}$, $\lambda = \lambda_0\delta^{1/2}$ with fixed $\lambda_0 < 0$ and $T \geqslant 0$ and we assume that $2 \leqslant |\log \delta|^7 \leqslant \Omega \leqslant \delta^{-1}$ We also fix $\beta > 0$, hence the limit of scaling parameters $\varepsilon = (\beta, \Omega, \lambda)$ is reduced to $\delta \to 0$, $\Omega \to \infty$ with the side condition that $\Omega \in [\,|\log \delta|^7, \delta^{-1}]$.

Let X be the solution to (4.36), then,

$$F(T\delta^{-1}) \leqslant C(\beta, \lambda_0, T)(1 + |x| + |v|) \tag{7.4}$$

where C is monotone increasing in T. Moreover, if $\widetilde{X}$ is the solution to (7.2), then the difference $Y(t) =: X(t) - \widetilde{X}(t)$ satisfies,

$$\lim_{\delta \to 0} \Big(\sup_{s \leqslant T\delta^{-1}} \mathbf{E}\,|Y(s)| + \sup_{s \leqslant T\delta^{-1}} \mathbf{E}\,|Y'(s)| \Big) = 0 \tag{7.5}$$

In particular,

$$\lim_{\delta \to 0} \int_{\mathbb{R}^2} \tilde{w}^{\varepsilon}_{A}(s, x, v)\, \bar{\phi}(x, v)\, dx\, dv = \lim_{\delta \to 0} \int_{\mathbb{R}^2} w^{\varepsilon}_{A}(s, x, v)\, \bar{\phi}(x, v)\, dx\, dv \tag{7.6}$$

uniformly for all $s \leqslant T\delta^{-1}$, where $\tilde{w}^{\varepsilon}_{A}(t, x, v)$ is the Wigner transform corresponding to $\tilde{X}$, defined exactly as (5.4), but $\tilde{X}(t)$ now being the solution to (7.2).

Proof. We follow essentially the proof of Lemma 4.1. The characteristics (4.36) fulfill

$$\begin{aligned} X(t) &= x \cos at + va^{-1} \sin at \\ &\quad + \int_0^t a^{-1} \sin a(t-s)[f(s) - (M \star X')(s) - xM(s)]\, ds \\ X'(t) &= -xa \sin at + v \cos at \\ &\quad + \int_0^t \cos a(t-s)[f(s) - (M \star X')(s) - xM(s)]\, ds \end{aligned} \tag{7.7}$$

Similarly to the proof of (4.30) one obtains

$$\mathbf{E} \left| \int_0^t a^{-1} \sin a(t-s)[(M \star X')(s) + xM(s)]\, ds \right| \leqslant K \left[\int_0^t F(s)\, ds + |x| \right] \tag{7.8}$$

recalling the value of K (4.25), and the cosine term in $X'(t)$ is similar.

Now we estimate the random forcing term. First we use

$$\mathbf{E} \left| \int_0^t f(s)\, a^{-1} \sin a(t-s)\, ds \right| \leqslant \left(\mathbf{E} \left| \int_0^t f(s)\, a^{-1} \sin a(t-s)\, ds \right|^2 \right)^{1/2} \tag{7.9}$$

then notice that $a^{-1} \sin a(t-s) = \eta_{\theta, \sigma}(t-s)$ with $\theta = 1$, $\sigma = 0$ (see (5.8)). Hence (cf. (5.12))

$$\mathbf{E} \left| \int_0^t f(s)\, a^{-1} \sin a(t-s)\, ds \right|^2 \leqslant \lambda^2 \int_0^{\Omega} A^2_{\beta}(\omega)\, H(t, \omega; 1, 0; a) \tag{7.10}$$

which is just $Q(t)=Q(t; 1, 0; \beta, a)$, see (5.11). Hence from (5.14), (5.15) we get

$$\mathbf{E}\left|\int_0^t f(s)\, a^{-1} \sin a(t-s)\, ds\right|^2 \leqslant C_1^2(\beta, \lambda_0, T) \tag{7.11}$$

using the relations among the parameters; $t=T\delta^{-1}$, $\lambda=\lambda_0\delta^{1/2}$ and $\Omega\leqslant\delta^{-1}$. Similar estimate is valid for the cosine term.

The estimates (7.8), (7.9) and (7.11) lead to the a priori bound,

$$F(t)\leqslant |x|+|v|+K\left[\int_0^t F(s)\, ds+|x|\right]+C_1(\beta, \lambda_0, T) \tag{7.12}$$

and by the standard Gronwall argument we obtain,

$$F(t)\leqslant C_2(\beta, \lambda_0, T)(1+|x|+|v|) \tag{7.13}$$

By monotonicity of C_2 in T, we get the a priori bound (7.4) on $X(t)$ and $X'(t)$.

From the equation (4.36) we also get a similar bound for $X''(t)$. We estimate

$$\mathbf{E}\,|X''(t)|\leqslant a^2\mathbf{E}\,|X(t)|+(\mathbf{E}\,|f(t)|^2)^{1/2}+|x|\,|M(t)|$$
$$+\int_0^t |M(s)|\;\mathbf{E}\,|X'(t-s)|\, ds$$

For the forcing term we use

$$\mathbf{E}\,|f(t)|^2=\lambda^2\int_0^\Omega \frac{\omega(\cosh\beta\omega+1)}{2\sinh\beta\omega}\, d\omega\leqslant C_3(\beta)\,\lambda^2\Omega^2$$

(see (4.32)) and that

$$|M(s)|=\lambda^2\left|\frac{\sin\Omega s}{s}\right|\leqslant\frac{2\Omega\lambda^2}{1+\Omega s} \tag{7.14}$$

These estimates, along with $t=T\delta^{-1}$, $\lambda=\lambda_0\delta^{1/2}$ and $\Omega\leqslant\delta^{-1}$, give that

$$\sup_{s\leqslant T\delta^{-1}}\mathbf{E}\,|X''(s)|\leqslant C_4(\beta, \lambda_0, T)(|x|+|v|+\Omega^{1/2}) \tag{7.15}$$

using the a priori bounds (4.37), and C_4 is monotone in T.

For the continuity result, notice that $Y(t) := X(t) - \widetilde{X}(t)$ satisfies the equation,

$$Y''(t) + I\lambda^2 Y'(t) + a^2 Y(t) = I\lambda^2 X'(t) - (M \star X')(t) - xM(t) \tag{7.16}$$

with initial conditions $Y(0) = Y'(0) = 0$. Using (7.3) we obtain,

$$\begin{aligned} &|I\lambda^2 X'(s) - (M \star X')(s)| \\ &\quad \leqslant \lambda^2 \left| \int_0^s \frac{\sin \Omega u}{u} (X'(s) - X'(s-u))\, du \right| + \lambda^2 \, |X'(s)| \left| \int_s^\infty \frac{\sin \Omega u}{u}\, du \right| \end{aligned} \tag{7.17}$$

The second term is estimated by $(const)\, \lambda^2 \, |X'(s)|$ with a universal constant if $s \leqslant 1$ and by $(const)\, \lambda^2 (\Omega s)^{-1} \, |X'(s)| \leqslant (const)\, \lambda^2 \Omega^{-1} \, |X'(s)|$ if $s \geqslant 1$.

In the first term we split the integration domain. For $u \geqslant \Omega^{-2/3}$ we use integration by parts, (4.37) and (7.15)

$$\begin{aligned} &\lambda^2 \mathbf{E} \left| \int_{\Omega^{-2/3}}^s \frac{d}{du} \left(\frac{\cos \Omega u}{\Omega} \right) u^{-1} (X'(s) - X'(s-u))\, du \right| \\ &\qquad \leqslant C_5(\beta, \lambda_0, T)\, \delta \, |\log \delta| \, \Omega^{-1/3} (1 + |x| + |v|) \end{aligned}$$

for all $s \leqslant T\delta^{-1}$. For the domain $0 \leqslant u \leqslant \Omega^{-2/3}$, we use Taylor expansion: $|X'(s) - X'(s-u)| \leqslant |u| \sup_{\sigma \leqslant s} |X''(\sigma)|$ and the bound (7.15). We obtain finally, using (4.37),

$$\mathbf{E}\, |I\lambda^2 X'(s) - (M \star X')(s)| \leqslant C_6(\beta, \lambda_0, T, x, v)\, \delta \, |\log \delta| \, \Omega^{-1/6} \tag{7.18}$$

if $1 \leqslant s \leqslant T\delta^{-1}$ and

$$\begin{aligned} &\mathbf{E}\, |I\lambda^2 X'(s) - (M \star X')(s)| \\ &\qquad \leqslant \pi \lambda_0^2 \delta F(t) \leqslant C_7(\beta, \lambda_0, x, v)\, \delta (1 + |\log \delta| \, \Omega^{-1/6}) \end{aligned} \tag{7.19}$$

if $s < 1$.

We now introduce the two fundamental solutions φ and ψ of $Y'' + I\lambda^2 Y' + a^2 Y = 0$ with $\varphi(0) = 0$, $\varphi'(0) = 1$ and $\psi(0) = 1$, $\psi'(0) = 0$. They are explicitly given as,

$$\varphi(t) = b^{-1} e^{-I\lambda^2 t/2} \sin bt, \qquad \psi(t) = e^{-I\lambda^2 t/2} \cos bt + \frac{I\lambda^2}{2} \varphi(t) \tag{7.20}$$

with $b := (a^2 - I^2\lambda^4/4)^{1/2}$. Note that they are bounded functions for small enough δ. Hence, by (7.14), (7.18) and (7.19),

$$\begin{aligned}
\mathbf{E}\,|Y(t)| &= \mathbf{E}\left|\int_0^t \varphi(t-s)(I\lambda^2 X'(s) - (M\star X')(s) - xM(s))\,ds\right| \\
&\leqslant (C_8(\beta,\lambda_0,T,x,v)\,|\log\delta|\,\Omega^{-1/6} + C_7(\beta,\lambda_0,x,v)\,\delta \\
&\quad + 2\lambda^2\,|x|\,[1+(\log\Omega t)_+])\,\|\phi\|_\infty \\
&\leqslant C_9(\beta,\lambda_0,T,x,v)\,\Omega^{-1/6}\,|\log\delta| \qquad (7.21)
\end{aligned}$$

The constants C_8 and C_9 can be chosen monotone in T, so the same estimate is valid for $\sup_{s\leqslant T\delta^{-1}}\mathbf{E}\,|Y(s)|$. The argument for Y' is similar, which proves (7.5). ∎

7.2. Transport Equation Before Scaling Limits

Armed with (7.6), it is enough to compute $\tilde{w}_A^\varepsilon(t,x,v)$. The calculation is the same as in Section 5.2 except for the different fundamental solutions φ and ψ given in (7.20). We redefine,

$$\begin{aligned}
\eta_{\theta,\sigma} &:= \theta\varphi(t) + \sigma\varphi'(t), \\
\xi_{\theta,\sigma} &:= \theta\psi(t) + \sigma\psi'(t)
\end{aligned} \qquad (7.22)$$

and in complete analogy to Lemma 5.2 we state the,

Lemma 7.2. We have for $t\geqslant 0$,

$$\int_{\mathbb{R}^2} \tilde{w}_A^\varepsilon(t,x,v)\,\overline{\phi(x,v)}\,dx\,dv$$

$$= \int_{\mathbb{R}^2} \hat{w}_0(\xi_{\theta,\sigma}(t),\eta_{\theta,\sigma}(t))\,\overline{\hat{\Phi}(\theta,\sigma)}\,e^{-(1/2)\,Q(t)}\,d\theta\,d\sigma \qquad (7.23)$$

with

$$Q(t) := \lambda^2\int_0^\Omega A_\beta^2(\omega)\,H(t,\omega)\,d\omega \qquad (7.24)$$

and H is given again as $H(t,\omega) = |\int_0^t \eta_{\theta,\sigma}(s)\,e^{-is\omega}\,ds|^2$, but with the new $\eta_{\theta,\sigma}$ defined in (7.22). We also have exactly the same estimate as (5.16), but with the redefined quantities. ∎

7.3. Obtaining Diffusion from Scaling Limit

In this section, and with similar arguments as in Section 6, we again obtain linear dependence in time of $Q(t)$ for large t. Indeed, we first write,

$$\varphi(t) = \frac{1}{2ib}(e^{tu} - e^{t\bar{u}}), \qquad \text{with} \quad u := -\frac{I\lambda^2}{2} + ib \tag{7.25}$$

With these notations, we have,

$$\eta_{\theta,\sigma}(t) = \frac{1}{2ib}(\theta(e^{tu} - e^{t\bar{u}}) + \sigma(ue^{tu} - \bar{u}e^{t\bar{u}})) \tag{7.26}$$

hence,

$$H(t,\omega) = \frac{1}{4b^2}\left|(\theta + \sigma u)\frac{e^{t(u-i\omega)} - 1}{u - i\omega} - (\theta + \sigma\bar{u})\frac{e^{t(\bar{u}-i\omega)} - 1}{\bar{u} - i\omega}\right|^2 \tag{7.27}$$

We now take the scaling $t = T\delta^{-1}$ for a fixed T and $\delta \to 0$. The terms with denominator $\bar{u} - i\omega = -I\lambda^2/2 - i(\sqrt{a^2 - I^2\lambda^4/4} + \omega)$ have no singularity (they are bounded) so the first term of H is the main term. Extracting the main term, we can write (cf. (5.20)),

$$H(t,\omega) = (\theta^2 + a^2\sigma^2)\left[\frac{1}{4a^2}\left|\frac{e^{t(u-i\omega)} - 1}{u - i\omega}\right|^2 + U(t,\omega)\right]$$

Using $u = ai + O(\delta)$, $0 < a^2 \leqslant 1$, $b^2 = a^2 + O(\delta^2)$ we obtain for small enough δ that,

$$\int_0^\infty |U(T\delta^{-1}, \omega)|\, d\omega \leqslant C_{10}(a, \beta, \lambda_0, T)\, |\log \delta|$$

With some elementary calculations this implies,

$$\begin{aligned}
Q(T\delta^{-1}) &= \lambda^2(\theta^2 + a^2\sigma^2)\left[\frac{1}{4a^2}\int_0^\Omega A_\beta^2(\omega)\left|\frac{e^{T\delta^{-1}(u-i\omega)} - 1}{u - i\omega}\right|^2 d\omega + B_1(T\delta^{-1})\right] \\
&= \lambda^2(\theta^2 + a^2\sigma^2)\left[\frac{A_\beta^2(a)}{4a^2}\int_{a+\sqrt{\delta}}^{a-\sqrt{\delta}}\left|\frac{e^{T\delta^{-1}(u-i\omega)} - 1}{u - i\omega}\right|^2 d\omega + B_2(T\delta^{-1})\right]
\end{aligned} \tag{7.28}$$

where the functions B_j $(j = 1, 2)$ satisfy $|B_j(T\delta^{-1})| \leqslant C_{11}(a, \beta, \lambda_0, T)\, \delta^{-1/2}$. We used that the function $\omega \mapsto A_\beta^2(\omega)$ is bounded with a bounded derivative around $\omega \sim a$, and that the function $z \mapsto (e^{tz} - 1)/z$ is uniformly bounded by t in the vicinity of the imaginary axis.

Since the derivative of $z \mapsto |(e^{tz} - 1)/z|^2$ is bounded by t^2, one can replace u by ai in the last integral at the expense of an error $2\sqrt{\delta}\,|u - ia|\, t^2 = O(\delta^{-1/2})$. Finally one can evaluate,

$$\int_{a+\sqrt{\delta}}^{a-\sqrt{\delta}} \left| \frac{e^{T\delta^{-1}(a-\omega)\,i} - 1}{a - \omega} \right|^2 d\omega = 2\pi T\delta^{-1} + O(\delta^{-1/2})$$

At this step $T \geqslant 0$ is used. In summary, we obtained,

$$Q(T\delta^{-1}) = (\theta^2 + a^2\sigma^2)\left(\lambda_0^2 T\, \frac{\pi(\cosh(\beta a) + 1)}{4a \sinh \beta a} + B_3(T\delta^{-1})\right) \tag{7.29}$$

The error satisfies $|B_3(T\delta^{-1})| \leqslant C_{12}(\beta, \lambda_0, T)\, \delta^{1/2}$, hence,

$$\lim_{\delta \to 0} Q(T\delta^{-1}) = c_\beta \lambda_0^2 \gamma^2 T \tag{7.30}$$

with $\gamma := \theta^2 + \check{a}^2\sigma^2$ and

$$c_\beta := \frac{\pi(\cosh(\beta\check{a}) + 1)}{4\check{a} \sinh \beta\check{a}} \tag{7.31}$$

assuming that

$$\check{a} := \lim_{\delta \to 0,\, \Omega \to \infty} a = \lim_{\delta \to 0,\, \Omega \to \infty} (1 - \lambda_0 \Omega \delta^{-1}) \tag{7.32}$$

exists, and $\check{a} \in (0, 1]$.

Since we will keep β fixed and choose $\lambda = \lambda_0 \delta^{1/2}$ with a fixed λ_0, δ and Ω are left as a scaling parameters from the triple $\varepsilon = (\beta, \Omega, \lambda)$. Like in Section 6 (cf. (6.2)) we introduce,

$$W_T^\varepsilon(x, v) := w_A^\varepsilon(T\delta^{-1}, x, v) \tag{7.33}$$

and notice that only the time is rescaled. We will assume that $\Omega \to \infty$ along with $\delta \to 0$ in such a way that the limit (7.32) exists and $\Omega \in [\,|\log \delta|^7, \delta^{-1}]$. Clearly either $\Omega \sim \delta^{-1}$, in which case $\check{a} < 1$, or $\Omega \ll \delta^{-1}$, when $\check{a} = 1$. In the latter case, however, we need $\Omega \geqslant |\log \delta|^7$.

7.4. Derivation of the Limiting Equation

We need the notion of "radial" function with respect to the elliptical phase space trajectories of the oscillator $Y'' + \check{a}^2 Y$. As usual, the dual variables to the phase space coordinates (x, v) are (θ, σ). With $\breve{a} > 0$ fixed, let

$$\gamma = \gamma(\theta, \sigma) := \sqrt{\theta^2 + \breve{a}^2\sigma^2}, \qquad r = r(x, v) := \sqrt{x^2 + \breve{a}^{-2}v^2}$$

which will be considered either variables or functions, depending on the context. If a function $u(x, v)$ depends only on $x^2 + \breve{a}^{-2}v^2$, then it can be written as $u(x, v) = u^*(r)$ with some function u^* defined on $\mathbb{R}_+$. Then the *two dimensional* Fourier transform $\hat{u}(\theta, \sigma) = \int \exp[-i(\theta x + \sigma v)]\, u(x, v)\, dx\, dv$ is a function of $\theta^2 + \breve{a}^2\sigma^2$ only, hence it can be written as $\hat{u}(\theta, \sigma) = \tilde{u}^*(\gamma)$. Here $\tilde{u}^*$ can be thought of as the "elliptical-radial" Fourier transform of u^*, but in order to avoid confusion, we will always perform Fourier transforms on $\mathbb{R}^2$, i.e., between $u(x, v) \leftrightarrow \hat{u}(\theta, \sigma)$, even if these functions are radial.

For any function $u(x, v)$ we can form the radial average of its Fourier transform $\hat{u}(\theta, \sigma)$ by defining

$$\hat{u}^{\#}(\theta, \sigma) := \frac{1}{2\pi} \int_0^{2\pi} \hat{u}(\gamma \cos s, \breve{a}^{-1}\gamma \sin s)\, ds$$

$$\left(= \frac{1}{2\pi\gamma} \int_{\tilde{\theta}^2 + \breve{a}^2\tilde{\sigma}^2 = \gamma^2} \hat{u}(\tilde{\theta}, \tilde{\sigma})\, d\tilde{\theta}\, d\tilde{\sigma} \right)$$

which is a function of γ, hence it can be written as

$$\hat{u}^{\#}(\theta, \sigma) = \tilde{u}^{\#,*}(\gamma)$$

In this notation $\#$ refers to radial averaging, and $*$ indicates that we consider the radial part of the function. Tilde indicates that it comes from the two dimensional Fourier transform $\hat{u}$ of the original function u.

Theorem 7.1. Define the large time scale Wigner function $W_T^\varepsilon(x, v)$ as in (7.33). Assume that $\lambda = \lambda_0 \delta^{1/2}$, $\lambda_0 < 1$ and fix $\beta > 0$, $\breve{a} \in (0, 1]$. The initial condition $W_0^\varepsilon(x, v) = w_0(x, v)$ satisfies $\hat{w}_0(\theta, \sigma) \in L^1(\mathbb{R}_\theta \times \mathbb{R}_\sigma)$. Consider the radial average of $\hat{W}_T^\varepsilon$,

$$\widetilde{W}_T^{\#,\varepsilon}(\gamma) := \frac{1}{2\pi} \int_0^{2\pi} \hat{W}_T^\varepsilon(\gamma \cos s, \breve{a}^{-1}\gamma \sin s)\, ds \tag{7.34}$$

Then for any $T \geq 0$ the limit,

$$\hat{W}_T^{+}(\theta, \sigma) := \lim_{\substack{\delta \to 0,\, \Omega \to \infty \\ 1 - \lambda_0^2 \Omega \delta \to \breve{a} \\ \Omega \geq |\log \delta|^7}} \widetilde{W}_T^{\#,\varepsilon}(\theta, \sigma) \tag{7.35}$$

exists in a weak sense and it is a function of $\gamma=(\theta^2+\breve{a}^2\sigma^2)^{1/2}$ only. Hence, its inverse Fourier transform $W_T^+(x,v)$ is a function of $r=(x^2+\breve{a}^{-2}v^2)^{1/2}$ only and it can be written as $W_T^{+,*}(r):=W_T^+(x,v)$. This function satisfies the radial Fokker–Planck equation,

$$\partial_T W_T^{+,*}=\frac{\pi\lambda_0^2}{4}\,\partial_r(rW_T^{+,*})+\frac{c_\beta\lambda_0^2}{2}\,\Delta_r W_T^{+,*} \tag{7.36}$$

(c_β is given in (7.31)) with initial condition $W_0^{+,*}(r):=W_{T=0}^+(x,v)$ whose Fourier transform $\hat{W}_0^+(\theta,\sigma)$ is given by,

$$\hat{W}_0^+(\theta,\sigma):=\hat{w}_0^\#(\theta,\sigma)=\frac{1}{2\pi}\int_0^{2\pi}\hat{w}_0(\gamma\cos s,\breve{a}^{-1}\gamma\sin s)\,ds \tag{7.37}$$

Remark 1. The weak limit $\lim^{**}\hat{W}_T^\varepsilon(\theta,\sigma)$ (without averaging over the angular variables) does not exist (here $\lim^{**}$ stands for the same limit as in (7.35)). However, time averaging can again replace angular averaging (see Remark and Corollary 6.1), i.e., our method easily proves that $\lim^{**}W_T^\varepsilon(x,v)$ exists in a weak sense in all variables (x,v,T), i.e., in the topology of $\mathscr{D}'(\mathbb{R}_x\times\mathbb{R}_v\times[0,\infty)_T)$, and it satisfies (7.36) weakly in space, velocity and time.

Remark 2. Since the diffusion coefficient $\frac{1}{2}\lambda_0^2 c_\beta$ in (7.36) behaves as β^{-1} for small β (high temperature), we see that Einstein's relation is satisfied at high temperatures. At small temperatures the diffusion does not disappear ($\lim_{\beta\to\infty}c_\beta>0$), which is due to the ground state quantum fluctuations of the heat bath.

Remark 3. Similarly to Remark 1 after Theorem 6.1, one can investigate how this theorem is modified if ϱ is not uniform (in particular if the cutoff is not sharp). The diffusive mechanism is not affected by this generalization, thanks to Remark 2 after Lemma 5.2, the only change is an extra $\varrho(\breve{a})$ factor in the second term on the right hand side of (7.36). But the modified memory kernel, $M(s)=\lambda^2\int_0^\Omega\cos(\omega s)\,\varrho(\omega)\,d\omega$, does not converge to the delta function $\delta_0(t)$ as $\Omega\to\infty$, hence the nonuniform frequency distribution makes the memory term nonlocal in time. The details are left to the reader.

Proof. The proof is similar to the proof of Theorem 6.1, hence we skip certain steps. Let $\phi(x,v)\in C_0^\infty(\mathbb{R}\times\mathbb{R})$. Similarly to (6.14) we obtain from (4.41),

$$\langle W^{\varepsilon}_{T}, \phi \rangle = \int \hat{w}^{\varepsilon}_{A}(T\delta^{-1}, \theta, \sigma)\, \overline{\hat{\phi}(\theta, \sigma)}\, d\theta\, d\sigma$$

$$= \mathbf{E} \int \hat{w}_0(\xi, \eta)\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{i(x\xi + v\eta)}\, e^{-i(\theta X(t) + \sigma X'(t))}\, d\xi\, d\eta\, dx\, dv\, d\theta\, d\sigma \tag{7.38}$$

Thanks to (7.6), in the limit $\delta \to 0$ we can replace X by $\tilde{X}$ and to take the limiting value (7.30) of Q in the formulae (we again have to approximate $\hat{w}_0$ by bounded functions first). We obtain (cf. (6.15)),

$$\lim{}^{**} \langle W^{\varepsilon}_{T}, \phi \rangle = \lim{}^{**} \mathbf{E} \int \hat{w}_0(\xi, \eta)\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{i(x\xi + v\eta)}$$

$$\times e^{-i(\theta \tilde{X}(T\delta^{-1}) + \sigma \tilde{X}'(T\delta^{-1}))}\, d\xi\, d\eta\, dx\, dv\, d\theta\, d\sigma$$

$$= \lim{}^{**} \int \hat{w}_0(\xi_{\theta, \sigma}(T\delta^{-1}), \eta_{\theta, \sigma}(T\delta^{-1}))$$

$$\times \overline{\hat{\phi}(\theta, \sigma)} e^{-(1/2)\, Q(T\delta^{-1})}\, d\theta\, d\sigma \tag{7.39}$$

where lim** stands for the limit in (7.35). Recall that the functions $\xi_{\theta, \sigma}$ and $\eta_{\theta, \sigma}$ now depend on the limiting parameters, since φ and ψ do, and they are oscillating, which again prevents the existence of the weak limit in the last line of (7.39) without averaging.

Time averaging is analogous to part (b) of Theorem 6.1, and it gives the weak limit in space, velocity and time. We skip the details of the proof of the statement of Remark 1.

Performing a radial avegaring (with respect to the limiting ellipses given by the level curves of $r = r(x, v)$ or $\gamma = \gamma(\theta, \eta)$) is the same as using radial testfunctions ϕ which depend only on r; i.e., $\hat{\phi}(\theta, \sigma)$ depends only on γ hence it can be written as $\hat{\phi}(\theta, \sigma) = \tilde{\phi}^*(\gamma)$. In this case

$$\langle \hat{W}^{\#, \varepsilon}_{T}, \hat{\phi} \rangle = \langle \hat{W}^{\varepsilon}_{T}, \hat{\phi} \rangle$$

From the explicit formulas (7.20), (7.22) it is straightforward to check that

$$\lim{}^{**} \sup_{s \leqslant T\delta^{-1}} |([\xi_{\theta, \sigma}(s)]^2 + \check{a}^2 [\eta_{\theta, \sigma}(s)]^2)$$

$$- e^{-I\lambda_0^2 s\delta}([\check{\xi}_{\theta, \sigma}(s)]^2 + \check{a}^2 [\check{\eta}_{\theta, \sigma}(s)]^2)| = 0 \tag{7.40}$$

where $\check{\xi}$ and $\check{\eta}$ are the solutions to $Y'' + \check{a}^2 Y = 0$, i.e.,

$$\check{\xi}_{\theta, \sigma}(s) := \theta \cos(\check{a}s) - \sigma \check{a} \sin(\check{a}s), \qquad \check{\eta}_{\theta, \sigma}(s) := \theta \check{a}^{-1} \sin(\check{a}s) + \sigma \cos(\check{a}s)$$

Since the flow $(\theta, \sigma) \mapsto (\xi_{\theta,\sigma}(s), \eta_{\theta,\sigma}(s))$ is measure preserving, one can change variables

$$\int_{\mathbb{R}^2} \hat{w}_0(\xi_{\theta,\sigma}(t), \eta_{\theta,\sigma}(t))\, \overline{\hat{\phi}(\theta, \sigma)}\, e^{-(1/2)\, Q(t)}\, d\theta\, d\sigma$$

$$= \int_{\mathbb{R}^2} \hat{w}_0(\theta, \sigma)\, \overline{\hat{\phi}(\xi^*_{\theta,\sigma}(t), \eta^*_{\theta,\sigma}(t))}\, e^{-(1/2)\, Q^*(t)}\, d\theta\, d\sigma \tag{7.41}$$

where $\eta^*(t) := \eta(-t)$, $\xi^*(t) := \xi(-t)$ are the backward trajectories. In this way we pushed the trajectories into the argument of $\hat{\phi}$, where only their $\xi^2 + \check{a}^2\eta^2$ combination matters, and we can apply (7.40) to replace ξ, η by $\check{\xi}, \check{\eta}$, finally we can change variables backwards, now along these new trajectories.

Hence together with (7.30) and with $c'_\beta := c_\beta/2$ for simplicity, we have

$$\begin{aligned} \lim{}^{**}\langle \hat{W}_T^{\#,\varepsilon}, \hat{\phi}\rangle &= \lim{}^{**}\langle \hat{W}_T^{\varepsilon}, \hat{\phi}\rangle \\ &= \lim{}^{**} \int_{\mathbb{R}^2} \hat{w}_0(e^{-I\lambda_0^2 T/2}\check{\xi}_{\theta,\sigma}(T\delta^{-1}), e^{-I\lambda_0^2 T/2}\check{\eta}_{\theta,\sigma}(T\delta^{-1})) \\ &\quad \times \overline{\tilde{\phi}^*(\gamma)}\, e^{-c'_\beta \lambda_0^2 T \gamma^2}\, d\theta\, d\sigma \end{aligned}$$

if we can show that this latter limit exists. But the right hand side above is in fact independent of the limiting parameters δ, Ω, since we can first integrate on ellipses $\theta^2 + \check{a}^2\sigma^2 = (const)$, similarly to the same calculation in the proof of part (c), Theorem 6.1. Hence,

$$\int_{\mathbb{R}^2} \hat{w}_0(e^{-I\lambda_0^2 T/2}\check{\xi}_{\theta,\sigma}(T\delta^{-1}), e^{-I\lambda_0^2 T/2}\check{\eta}_{\theta,\sigma}(T\delta^{-1}))\, \overline{\tilde{\phi}^*(\gamma)}\, e^{-c'_\beta \lambda_0^2 T \gamma^2}\, d\theta\, d\sigma$$

$$= \int_{\mathbb{R}^2} \tilde{W}_0^{+,*}(\gamma e^{-I\lambda_0^2 T/2})\, \overline{\tilde{\phi}^*(\gamma)}\, e^{-c'_\beta \lambda_0^2 T\gamma^2}\, d\theta\, d\sigma \tag{7.42}$$

where we recall the definition of $\tilde{W}_0^+$ (7.37), which depends only on $\gamma^2 = \theta^2 + \check{a}^2\sigma^2$, and we let $\tilde{W}_0^{+,*}(\gamma) := \tilde{W}_0^+(\theta, \sigma)$. Therefore, the relation,

$$\lim{}^{**}\langle \hat{W}_T^{\#,\varepsilon}, \hat{\phi}\rangle = \int_{\mathbb{R}^2} \tilde{W}_0^{+,*}(\gamma e^{-I\lambda_0^2 T/2})\, \overline{\tilde{\phi}^*(\gamma)}\, e^{-c'_\beta \lambda_0^2 T\gamma^2}\, d\theta\, d\sigma$$

defines the weak limit,

$$\hat{W}_T^+(\theta, \sigma) := \lim{}^{**}\hat{W}_T^{\#,\varepsilon}(\theta, \sigma)$$

and it is a function depending only on $\theta^2+\check{\alpha}^2\sigma^2$, i.e., it can be written as $\tilde{W}_T^{+,*}(\gamma) := \hat{W}_T^{+}(\theta,\sigma)$. Also, we readily obtain the equation satisfied by $\tilde{W}_T^{+,*}(\gamma)$ by computing,

$$\begin{aligned}\langle \partial_T|_{T=0}\hat{W}_T^{+},\hat{\phi}\rangle &= \partial_T|_{T=0}\int_{\mathbb{R}^2}\hat{W}_0^{+,*}(\gamma e^{-I\lambda_0^2T/2})\,\overline{\tilde{\phi}^*(\gamma)}\,e^{-c'_\beta\lambda_0^2T\gamma^2}\,d\theta\,d\sigma\\ &= \int_{\mathbb{R}^2}\left[-\frac{I\lambda_0^2}{2}\,\gamma\partial_\gamma - c'_\beta\lambda_0^2\gamma^2\right]\tilde{W}_0^{+,*}(\gamma)\,\overline{\tilde{\phi}^*(\gamma)}\,d\theta\,d\sigma \qquad (7.43)\end{aligned}$$

from which (7.36) follows, recalling that $I=\pi/2$ and the value of $c'_\beta=c_\beta/2$ from (7.31). ∎

ACKNOWLEDGMENTS

The authors are indebted to H. Spohn for discussions. F.C. and L.E. were partially supported by the Erwin Schrödinger Institute in Vienna (Austria) during their visit, and they thank this institution for its hospitality. This work was supported by the TMR-Network "Asymptotic Methods in Kinetic Theory" number ERB FMBX CT97 0157 (F.C., F.F., and P.A.M.) and by NSF Grant DMS-9970323 (L.E.).

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